

Normal approximation of Gibbsian sums in geometric probability

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September 24, 2014

Abstract

This paper concerns the asymptotic behavior of a random variable W_λ resulting from the summation of the functionals of a Gibbsian spatial point process over windows $Q_\lambda \uparrow \mathbb{R}^d$. We establish conditions ensuring that W_λ has volume order fluctuations, that is they coincide with the fluctuations of functionals of Poisson spatial point processes. We combine this result with Stein's method to deduce rates of normal approximation for W_λ , as $\lambda \rightarrow \infty$. Our general results establish variance asymptotics and central limit theorems for statistics of random geometric and related Euclidean graphs on Gibbsian input. We also establish similar limit theory for claim sizes of insurance models with Gibbsian input, the number of maximal points of a Gibbsian sample, and the size of spatial birth-growth models with Gibbsian input.

Key words and phrases. Gibbs point process, Berry–Esseen bound, Stein's method, random Euclidean graphs, maximal points, spatial birth-growth models.

AMS 2010 Subject Classification: Primary 60F05; secondary 60D05, 60G55.

1 Introduction and main results

Functionals of large geometric structures on finite input $\mathcal{X} \subset \mathbb{R}^d$ often consist of sums of spatially dependent terms admitting the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1.1}$$

Research supported in part by ARC Discovery Grant DP130101123 (AX) and NSF grants DMS-1106619, DMS-1406410 (JY)

where the \mathbb{R}^+ -valued *score function* ξ , defined on pairs (x, \mathcal{X}) , represents the *interaction* of x with respect to \mathcal{X} . The sums (1.1) typically describe a global feature of an underlying geometric property in terms of a sum of local contributions $\xi(x, \mathcal{X})$.

A large and diverse number of functionals and statistics in stochastic geometry, applied geometric probability, and spatial statistics may be cast in the form (1.1) for appropriately chosen ξ . The behavior of these statistics on random input \mathcal{X} can be deduced from general limit theorems [5, 27, 28, 31, 32] for (1.1) provided \mathcal{X} is either a Poisson or binomial point process. This has led to solutions of problems in random sequential packing [30], random graphs [27, 28, 29, 31, 36], percolation models [20], analysis of data on manifolds [33], and convex hulls of i.i.d. samples [7, 8, 9], among others.

When \mathcal{X} is neither Poisson nor binomial input, the limit theory of (1.1) is less well understood. Our main purpose is to redress this for Gibbsian input. For all $\lambda \in [1, \infty)$ consider the functionals

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}),$$

where $\mathcal{P}_\lambda^{\beta\Psi}$ is the restriction of a Gibbs point process $\mathcal{P}^{\beta\Psi}$ on \mathbb{R}^d to $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$. The process $\mathcal{P}^{\beta\Psi}$ has potential Ψ , it is absolutely continuous with respect to a reference homogeneous Poisson point process $\tilde{\mathcal{P}}_\tau$ of intensity τ , and β is the inverse temperature. In general, even for the simplest of score functions ξ , the Gibbsian functional W_λ may neither enjoy asymptotic normality nor have volume order fluctuations, i.e., $\text{Var} W_\lambda$ may not be of order $\text{Vol}(Q_\lambda)$; see [21]. On the other hand, if both the Gibbsian input and the score function have rapidly decaying spatial dependencies, then one could expect that W_λ behaves like a sum of i.i.d. random variables.

We have three goals. The first is to show that given a potential Ψ , there is a range of inverse temperature and intensity parameters β and τ such that for any locally determined score function, the Gibbsian functional W_λ has volume order fluctuations. In other words, the fluctuations for W_λ coincide with those when $\mathcal{P}_\lambda^{\beta\Psi}$ is replaced by Poisson or binomial input. This strengthens the central limit theorems of [35], which depend crucially on volume order fluctuations. Our second goal is to prove a rate of convergence to the normal for $(W_\lambda - \mathbb{E} W_\lambda)/\sqrt{\text{Var} W_\lambda}$ for general score functions ξ , including those which are non-translation invariant. Formal statements of these results are given in Theorems 1.1-1.3. Thirdly, we use our general results to deduce rates of normal convergence for (i) statistics of random geometric and Euclidean graphs on Gibbsian input, (ii) the number of claims in an insurance model with claim locations and times given by Gibbsian input, (iii) the number of maximal points in a Gibbs sample, as well as (iv) functionals of spatial birth-growth models with Gibbsian input. This extends the central limit theorems and second order results of [4, 19, 27, 31, 32] to Gibbsian input.

1.1 Notation and terminology

(i) **Gibbs point processes.** Quantifying spatial dependencies of Gibbs point processes is difficult in general. However spatial dependencies readily become transparent when a Gibbs point process is viewed as an algorithmic construct. As shown in [35], this is feasible whenever Ψ belongs to the class of potentials Ψ^* containing pair potentials, continuum Widom-Rowlinson potentials, area interaction potentials, hard core potentials and potentials generating a truncated Poisson point process.

We review the algorithmic construction of Gibbs point processes developed in [35], and inspired by [16]. Define for $\Psi \in \Psi^*$ and finite $\mathcal{X} \subset \mathbb{R}^d$ the local energy function

$$\Delta^\Psi(\mathbf{0}, \mathcal{X}) := \Psi(\mathcal{X} \cup \{\mathbf{0}\}) - \Psi(\mathcal{X}), \quad \mathbf{0} \notin \mathcal{X}.$$

Here $\mathbf{0}$ denotes a point at the origin of \mathbb{R}^d . Proposition 2.1 (i) of [35] shows that for $\mathcal{X} \subset \mathbb{R}^d$ locally finite,

$$\Delta^\Psi(\mathbf{0}, \mathcal{X}) := \lim_{r \rightarrow \infty} \Delta^\Psi(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})) \quad (1.2)$$

is well-defined, where $B_r(x) := \{y : |x - y| \leq r\}$ is the Euclidean ball with center x and radius r . Ψ has *finite or bounded range* if there is $r^\Psi \in (0, \infty)$ such that for all finite $\mathcal{X} \subset \mathbb{R}^d$ we have $\Delta^\Psi(\mathbf{0}, \mathcal{X}) = \Delta^\Psi(\mathbf{0}, \mathcal{X} \cap B_{r^\Psi}(\mathbf{0}))$. With the exception of the pair potential, all potentials in Ψ^* have finite range (Lemma 3.1 of [35]). For such Ψ we put

$$m_0^\Psi := \inf_{\mathcal{X} \text{ locally finite}} \Delta^\Psi(\mathbf{0}, \mathcal{X})$$

and

$$\mathcal{R}^\Psi := \{(u, v) \in (\mathbb{R}^+)^2 : uv_d \exp(-vm_0^\Psi)(r^\Psi + 1)^d < 1\}, \quad (1.3)$$

where $v_d := \pi^{d/2}[\Gamma(1 + d/2)]^{-1}$ is the volume of the unit ball in \mathbb{R}^d . When Ψ is a pair potential, then the factor $(r^\Psi + 1)^d$ in (1.3) is replaced by the moment of an exponentially decaying random variable as in (3.7) of [35].

Let $(\varrho(t))_{t \in \mathbb{R}}$ be a stationary homogeneous free birth and death process on \mathbb{R}^d with these dynamics:

- A new point $x \in \mathbb{R}^d$ is born in ϱ_t during the time interval $[t - dt, t]$ with probability $\tau dx dt$,
- An existing point $x \in \varrho_t$ dies during the time interval $[t - dt, t]$ with probability dt , that is the lifetimes of points of the process are independent standard exponential.

The unique stationary and reversible measure for this process is the law of the Poisson point process $\tilde{\mathcal{P}}_\tau$.

Following [35], for each $\Psi \in \Psi^*$, we use a dependent thinning procedure on $(\varrho(t))_{t \in \mathbb{R}}$ to algorithmically construct a Gibbs point process $\mathcal{P}^{\beta\Psi}$ on \mathbb{R}^d , one whose law is absolutely

continuous with respect to the reference point process $\tilde{\mathcal{P}}_\tau$. Section 3 recalls some of the salient properties of $\mathcal{P}^{\beta\Psi}$.

For arbitrary (τ, β) and arbitrary Ψ , the asymptotic behavior of W_λ may involve non-standard scaling and non-standard limits. However, if $\mathcal{P}^{\beta\Psi}$ is *admissible in the sense that* $(\tau, \beta) \in \mathcal{R}^\Psi$ and $\Psi \in \Psi^*$, then we shall show that W_λ behaves like a classical sum of i.i.d. random variables. Henceforth, and without further mention, we shall always assume that $\mathcal{P}^{\beta\Psi}$ is admissible. Recall that $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$ and put $Q_\infty := \mathbb{R}^d$. Given $\lambda \in [1, \infty]$, $\Psi \in \Psi^*$, and $(\tau, \beta) \in \mathcal{R}^\Psi$, we let

$$\mathcal{P}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap Q_\lambda. \quad (1.4)$$

By convention we have $\mathcal{P}_\infty^{\beta\Psi} := \mathcal{P}^{\beta\Psi}$.

(ii) **Poisson-like point processes.** A point process Ξ on \mathbb{R}^d is *stochastically dominated* by the reference process $\tilde{\mathcal{P}}_\tau$ if for all Borel sets $B \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ we have $\mathbb{P}[\text{card}(\Xi \cap B) \geq n] \leq \mathbb{P}[\text{card}(\tilde{\mathcal{P}}_\tau \cap B) \geq n]$. As in [35], we say that Ξ is *Poisson-like* if (a) Ξ is stochastically dominated by $\tilde{\mathcal{P}}_\tau$ and (b) there exists $c \in (0, \infty)$ and $r_1 \in (0, \infty)$ such that for all $r \in (r_1, \infty)$, $x \in \mathbb{R}^d$, and any point set $\mathcal{E}_r(x)$ in $B_r^c(x)$, the conditional probability of $B_r(x)$ not being hit by Ξ , given that $\Xi \cap B_r(x)^c$ coincides with $\mathcal{E}_r(x)$, satisfies

$$\mathbb{P}[\Xi \cap B_r(x) = \emptyset \mid \{\Xi \cap B_r(x)^c = \mathcal{E}_r(x)\}] \leq \exp(-cr^d). \quad (1.5)$$

Poisson-like processes have void probabilities analogous to those of homogeneous Poisson processes, justifying the choice of terminology. Lemma 3.3 of [35] shows that admissible Gibbs processes $\mathcal{P}^{\beta\Psi}$ are Poisson-like.

(iii) **Translation invariance.** ξ is *translation invariant* if for all $x \in \mathbb{R}^d$ and locally finite $\mathcal{X} \subset \mathbb{R}^d$ we have $\xi(x, \mathcal{X}) = \xi(x + y, \mathcal{X} + y)$ for all $y \in \mathbb{R}^d$.

(iv) **Moment conditions.** Let $\|X\|_q$ denote the q norm of the random variable X . Say that ξ satisfies the q -moment condition if

$$w_q := \sup_{\lambda \in [1, \infty]} \sup_{x \in Q_\lambda} \|\xi(x, \mathcal{P}_\lambda^{\beta\Psi} \cup \{x\})\|_q < \infty. \quad (1.6)$$

(v) **Stabilization.** Given a locally finite point set \mathcal{X} , write \mathcal{X}^z for $\mathcal{X} \cup \{z\}$ if $z \in \mathbb{R}^d$ and $\mathcal{X}^z = \mathcal{X}$ if $z = \emptyset$. The following definition of stabilization is similar to that in [3, 27, 28, 31, 32] except now we consider Gibbsian input, instead of Poisson or binomial input.

Definition 1.1 ξ is a stabilizing functional with respect to the Poisson-like process Ξ if for all $x \in \mathbb{R}^d$, all $z \in \mathbb{R}^d \cup \{\emptyset\}$, and almost all realizations \mathcal{X} of Ξ there exists $R := R^\xi(x, \mathcal{X}^z) \in (0, \infty)$ (a ‘radius of stabilization’) such that

$$\xi(x, \mathcal{X}^z \cap B_R(x)) = \xi(x, (\mathcal{X}^z \cap B_R(x)) \cup \mathcal{Y}) \quad (1.7)$$

for all locally finite point sets $\mathcal{Y} \subseteq \mathbb{R}^d \setminus B_R(x)$.

Stabilization of ξ on Ξ implies that $\xi(x, \mathcal{X}^z)$ is wholly determined by the point configuration $\mathcal{X}^z \cap B_{R^\xi}(x)$. It also yields $\xi(x, \mathcal{X}^z \cap B_r(x)) = \xi(x, \mathcal{X}^z \cap B_{R^\xi}(x))$ for $r \in [R^\xi, \infty)$. Stabilizing functionals can thus be a.s. extended to the entire process Ξ^z , that is to say for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \cup \{\emptyset\}$ we have

$$\xi(x, \Xi^z) := \lim_{r \rightarrow \infty} \xi(x, \Xi^z \cap B_r(x)) \quad \text{a.s.} \quad (1.8)$$

Given $s > 0$ and any simple point process Ξ , including Poisson-like processes, define the conditional tail probability

$$t(\Xi, s) := \sup_{x \in \mathbb{R}^d} \sup_{z \in \mathbb{R}^d \cup \{\emptyset\}} \mathbb{P}[R^\xi(x, \Xi^z) > s | \Xi\{x\} = 1].$$

The conditional distribution of Ξ given that $\Xi\{x\} = 1$ is the Palm distribution of Ξ at x [18, Chapter 10] and the conditional probability can be intuitively interpreted as

$$\mathbb{P}[R^\xi(x, \Xi^z) > s | \Xi\{x\} = 1] = \lim_{\epsilon \downarrow 0} \mathbb{P}\left[\sup_{y \in B_\epsilon(x) \cap \Xi} R^\xi(y, \Xi^z) > s | \Xi(B_\epsilon(x)) = 1\right].$$

We say that ξ is *stabilizing in the wide sense* if for every Poisson-like process Ξ we have $t(\Xi, s) \rightarrow 0$ as $s \rightarrow \infty$. Further, ξ is *exponentially stabilizing in the wide sense* if for every Poisson-like process Ξ we have

$$\limsup_{s \rightarrow \infty} s^{-1} \ln t(\Xi, s) < 0. \quad (1.9)$$

Exponential stabilization of ξ with respect to the augmented point set Ξ^z ensures that covariances of scores at points x and y , as given at (1.15), decays exponentially fast with $|x - y|$, implying that W_λ has at most volume order fluctuations, as seen in the proof of Lemma 4.6. Notice that for λ large we have $R^\xi(x, \Xi^z \cap Q_\lambda) \leq R^\xi(x, \Xi^z)$ and thus (1.9) holds with $t(\Xi, s)$ replaced by

$$\limsup_{\lambda \rightarrow \infty} \sup_{x \in Q_\lambda} \sup_{z \in \mathbb{R}^d \cup \{\emptyset\}} \mathbb{P}[R^\xi(x, \Xi^z \cap Q_\lambda) > s | \Xi\{x\} = 1]. \quad (1.10)$$

For a set $E \subset \mathbb{R}^d$, let $\text{Vol}_d(E)$ denote the d -dimensional volume of E . For $u \in (0, \infty)$, we let $Q_u \subset \mathbb{R}^d$ be the cube centered at the origin having $\text{Vol}_d(Q_u) = u$.

(vi) **Non-degeneracy with respect to $\mathcal{P}^{\beta\Psi}$.** Say that ξ satisfies non-degeneracy with respect to $\mathcal{P}^{\beta\Psi}$ if there exists $r \in (0, \infty)$ and $b_0 := b_0(r) \in (0, \infty)$ such that given $\mathcal{P}^{\beta\Psi} \cap Q_r^c$, the sum $\sum_{x \in \mathcal{P}^{\beta\Psi} \cap Q_t} \xi(x, \mathcal{P}^{\beta\Psi})$ has expected variability bounded below by b_0 , uniformly in $t \in [r, \infty)$. In other words, we have

$$\inf_{t \in [r, \infty)} \mathbb{E} \text{Var} \left[\sum_{x \in \mathcal{P}^{\beta\Psi} \cap Q_t} \xi(x, \mathcal{P}^{\beta\Psi}) \mid \mathcal{P}^{\beta\Psi} \cap Q_r^c \right] \geq b_0. \quad (1.11)$$

As shown in Section 2, functionals of interest often satisfy (1.11). There is nothing special about using cubes Q_r in (1.11) and, as can be seen from the proofs, Q_r could be replaced by any compact convex subset of \mathbb{R}^d .

If f and g are two functions satisfying $\liminf_{\lambda \rightarrow \infty} f(\lambda)/g(\lambda) > 0$ then we write $f(\lambda) = \Omega(g(\lambda))$. If, in addition we have $f(\lambda) = O(g(\lambda))$ then we write $f(\lambda) = \Theta(g(\lambda))$.

From the standpoint of applications, it is useful to have a version of (1.11) for score functions which are not translation invariant and for input

$$\tilde{\mathcal{P}}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_\lambda, \quad (1.12)$$

where $\tilde{S}_\lambda \subset \mathbb{R}^d$ satisfies $\text{Vol}_d(\tilde{S}_\lambda) = \Omega(1)$. Here and elsewhere $\tilde{Q}_u \subset \mathbb{R}^d$ denotes a cube with $\text{Vol}_d(\tilde{Q}_u) = u$, but not necessarily centered at the origin.

(vii) **Non-degeneracy with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$.** Say that ξ satisfies non-degeneracy with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ if there is $r \in (0, \infty)$ and $b_0 := b_0(r) \in (0, \infty)$, such that for λ large there is $\tilde{Q}_r \subset \tilde{S}_\lambda$ satisfying

$$\mathbb{E} \text{Var} \left[\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}) \mid \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r^c \right] \geq b_0. \quad (1.13)$$

Given $\rho \in (r, \infty)$, let $\mathcal{C}(\rho, r, \tilde{S}_\lambda)$, be a collection of d -dimensional volume r cubes $\tilde{Q}_{i,r}$, $1 \leq i \leq n(\rho, r, \tilde{S}_\lambda)$, which are separated by 4ρ and which satisfy (1.13).

For all x and y in \mathbb{R}^d we put

$$c^\xi(x) := \mathbb{E} \xi(x, \mathcal{P}^{\beta\Psi}) \exp(-\beta\Delta(x, \mathcal{P}^{\beta\Psi})), \quad (1.14)$$

and

$$c^\xi(x, y) := c^\xi(x)c^\xi(y) - \mathbb{E} \xi(x, \mathcal{P}^{\beta\Psi} \cup \{y\}) \xi(y, \mathcal{P}^{\beta\Psi} \cup \{x\}) \cdot \exp(-\beta\Delta(\{x, y\}, \mathcal{P}^{\beta\Psi})). \quad (1.15)$$

Put

$$\sigma^2(\xi, \tau) := c^{\xi^2}(\mathbf{0}) - \tau \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy. \quad (1.16)$$

1.2 Main results

The following are our main results. Applications follow in Section 2. Our first result gives conditions under which the Gibbsian functional W_λ has volume order fluctuations.

Theorem 1.1 *Assume that ξ is translation invariant, exponentially stabilizing in the wide sense (1.9) and satisfies the q -moment condition (1.6) for some $q \in (2, \infty)$. Then*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var} W_\lambda = \tau \sigma^2(\xi, \tau) \in [0, \infty). \quad (1.17)$$

If, in addition, ξ satisfies non-degeneracy (1.11), then $\sigma^2(\xi, \tau) > 0$.

Recall that the Kolmogorov distance between the distributions of random variables X_1 and X_2 is defined as

$$d_K(X_1, X_2) := \sup_{t \in \mathbb{R}} |\mathbb{P}[X_1 \leq t] - \mathbb{P}[X_2 \leq t]|.$$

Theorem 1.2 *Assume that ξ is exponentially stabilizing in the wide sense (1.9) and satisfies the q -moment condition (1.6) for some $q \in (2, \infty)$. For all $p \in (2, q)$, put $p_3 := p_3(p) := \min\{p, 3\}$. Then*

$$d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1) \right) = O((\ln \lambda)^{d(p_3-1)} \lambda (\text{Var} W_\lambda)^{-p_3/2}). \quad (1.18)$$

Furthermore, if ξ is translation invariant, satisfies non-degeneracy (1.11) and the q -moment condition (1.6) for some $q \in (3, \infty)$, then

$$d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}) \quad (1.19)$$

and therefore as $\lambda \rightarrow \infty$

$$\lambda^{-1/2} (W_\lambda - \mathbb{E} W_\lambda) \xrightarrow{\mathcal{D}} N(0, \tau \sigma^2(\xi, \tau)).$$

Remarks. (i) (Theorem 1.1.) The proof of volume order variance asymptotics is indirect. We first show that $\text{Var} W_\lambda$ is of volume order up to a logarithmic term (Lemma 4.3). Putting $\hat{W}_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\})$ we then show in Lemma 4.6 the dichotomy that either $\text{Var} \hat{W}_\lambda = \Omega(\lambda)$ or $\text{Var} \hat{W}_\lambda = O(\lambda^{(d-1)/d})$. Closeness of $\text{Var} W_\lambda$ and $\text{Var} \hat{W}_\lambda$, as shown in Lemma 4.5, completes the argument, whose full details are in Section 3. Under condition (1.11) we obtain volume order variance asymptotics when $\mathcal{P}^{\beta\Psi}$ is replaced by a homogeneous Poisson point process, which is of independent interest. Verifying condition (1.11) for Gibbsian input is comparable to verifying the non-degeneracy conditions of Theorem 2.1 of [29] or Theorem 1.2 of [14]

(ii) (Theorem 1.2.) Theorem 2.3 of [35] shows the rate of convergence $O((\ln \lambda)^{3d} \lambda^{-1/2})$ in (1.19). However this result assumes that $\text{Var} W_\lambda = \Theta(\lambda)$, which may not always hold, particularly when the scaling is not volume order. Theorem 1.2 contains no such assumption. Theorem 1.2 extends Corollary 3.1 of [4] to Gibbsian input. We do not take up the question of laws of large numbers for W_λ as this is addressed in [35].

(iii) (Point processes with marks.) Let $(\mathcal{E}, \mathcal{F}_\mathcal{E}, \mu_\mathcal{E})$ be a probability space (the mark space) and consider the marked reference Poisson point process $\{(x, a); x \in \tilde{\mathcal{P}}_\tau, a \in \mathcal{E}\}$ in the space $\mathbb{R}^d \times \mathcal{E}$ with law given by the product measure of the law of $\tilde{\mathcal{P}}_\tau$ and $\mu_\mathcal{E}$. Then the proofs of Theorems 1.1 and 1.2 go through in this setting, where it is understood that in the algorithmic construction the process $\mathcal{P}_\lambda^{\beta\Psi}$ inherits the marks from $\tilde{\mathcal{P}}_\tau$ and where the cubes Q_r in condition (1.11) are replaced with cylinders $C_r := Q_r \times \mathcal{E}$. This generalization is used in Section 2.5 to deduce central limit theorems for spatial birth-growth models with Gibbsian input.

Next we consider the analog of W_λ on input $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ defined at (1.12), namely

$$\tilde{W}_\lambda := \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}).$$

Say that ξ satisfies the q -moment condition with respect to $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ if

$$\sup_{\lambda \in [1, \infty)} \sup_{x \in \tilde{S}_\lambda} \|\xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cup \{x\})\|_q < \infty. \quad (1.20)$$

The following result does not assume that ξ is translation invariant.

Theorem 1.3 *Assume that ξ is exponentially stabilizing in the wide sense (1.9) and satisfies the q -moment condition (1.20) for some $q \in (2, \infty)$. For all $p \in (2, q)$, put $p_3 := p_3(p) := \min\{p, 3\}$. Then*

$$d_K \left(\frac{\tilde{W}_\lambda - \mathbb{E} \tilde{W}_\lambda}{\sqrt{\text{Var} \tilde{W}_\lambda}}, N(0, 1) \right) = O \left((\ln \lambda)^{d(p_3-1)} \text{Vol}(\tilde{S}_\lambda) (\text{Var} \tilde{W}_\lambda)^{-p_3/2} \right). \quad (1.21)$$

Furthermore, if ξ satisfies non-degeneracy (1.13) and $\rho \in (c \ln \lambda, \infty)$, c large, then

$$\text{Var} \tilde{W}_\lambda \geq c^{-1} b_0 n(\rho, r, \tilde{S}_\lambda). \quad (1.22)$$

If $q \in (3, \infty)$ we thus have

$$d_K \left(\frac{\tilde{W}_\lambda - \mathbb{E} \tilde{W}_\lambda}{\sqrt{\text{Var} \tilde{W}_\lambda}}, N(0, 1) \right) = O \left((\ln \lambda)^{2d} \text{Vol}(\tilde{S}_\lambda) n(\rho, r, \tilde{S}_\lambda)^{-3/2} \right). \quad (1.23)$$

Remark. The bound (1.22) shows volume order growth for $\text{Var}\tilde{W}_\lambda$, but only up to the logarithmic factor $(\ln \lambda)^d$. When ξ is translation invariant we are able to remove this factor, as described in Remark (i) following Theorem 1.2. However for non-translation invariant ξ , we are unable to remove the logarithmic factor. Consequently, the bound (1.19) is smaller than the bound (1.23) by a factor $((\ln \lambda)^d)^{3/2}$.

2 Applications

We deduce variance asymptotics and central limit theorems for six well-studied functionals in geometric probability. Save for some special cases as noted below, the limit theory for these functionals has, up to now, been largely confined to Poisson or binomial input. Our examples are not exhaustive. For example, there is scope for treating the limit theory of coverage processes on Gibbsian input, and, more generally, the limit theory of functionals of germ-grain models, with germs given by the realization of $\mathcal{P}^{\beta\Psi}$. One could also treat the limit theory of functionals arising in percolation and nucleation models having Gibbsian input, extending [20] and [17], respectively.

2.1 Clique counts in random geometric graphs

Let $\mathcal{X} \subset \mathbb{R}^d$ be locally finite and put $s \in (0, \infty)$. The geometric graph on \mathcal{X} , here denoted $GG_s(\mathcal{X})$, is obtained by connecting points $x, y \in \mathcal{X}$ with an edge whenever $|x - y| \leq s$. If there is a subset $S := S(s, k)$ of \mathcal{X} of size $k + 1$ with all points of S within a distance s of each other, then the k simplex formed by S has edges in $GG_s(\mathcal{X})$. The Vietoris-Rips complex $\mathcal{R}^s(\mathcal{X})$, or Rips complex, is the simplicial complex arising as the union of all k -simplices $S(s, k) \subset GG_s(\mathcal{X})$. The Vietoris-Rips complex and the closely related Čech complex (which has a simplex for every finite subset of balls in $GG_s(\mathcal{X})$ with non-empty intersection) are used to model the topology of ad hoc sensor and wireless networks and they are also useful in the statistical analysis of high-dimensional data sets. Note that $\mathcal{C}_k^s(\mathcal{X})$ is the number of cliques of order $k + 1$ in $GG_s(\mathcal{X})$. For \mathcal{X} random, the number $\mathcal{C}_k^s(\mathcal{X})$ of k -simplices in $GG_s(\mathcal{X})$ is of theoretical and applied interest (see e.g. [26]). The limit theory for $\mathcal{C}_k^s(\mathcal{X})$ is well understood when \mathcal{X} is Poisson or binomial input on \mathbb{R}^d [26] or on a manifold [33]. We are unaware of limit theory for $\mathcal{C}_k^s(\cdot)$ on Gibbsian input. For all $k = 1, 2, \dots$ and all $s \in (0, \infty)$ let $\xi_k(x, \mathcal{X}) := \xi_k^{(s)}(x, \mathcal{X})$ be $(k + 1)^{-1}$ times the number of k -simplices in $\mathcal{R}^s(\mathcal{X})$ containing the vertex x .

Theorem 2.1 For all $k = 1, 2, \dots$ and all $s \in (0, \infty)$ we have

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var}[\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})] = \tau \sigma^2(\xi_k, \tau) > 0,$$

and

$$d_K \left(\frac{\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi}) - \mathbb{E} \mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})}{\sqrt{\text{Var}[\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi})]}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Proof. We have $\mathcal{C}_k^s(\mathcal{P}_\lambda^{\beta\Psi}) = \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_k(x, \mathcal{P}_\lambda^{\beta\Psi})$. It suffices to show that ξ_k satisfies the conditions of Theorems 1.1 and 1.2. Given $x \in \mathbb{R}^d$ and $k = 1, 2, \dots$ we note that $\xi_k(x, \mathcal{P}_\lambda^{\beta\Psi})$ is generously bounded by $\left(\sum_{X_i \in \mathcal{P}_\lambda^{\beta\Psi}} \mathbf{1}(|x - X_i| \leq s) \right)^k$, which is in turn bounded by the k th power of a Poisson random variable with parameter $\tau \text{Vol}_d(B_s(x))$. Since all moments of Poisson random variables are finite, it follows that ξ_k satisfies the moment condition (1.6) for all $q \in (1, \infty)$. Clearly ξ_k is translation invariant and exponentially stabilizing with stabilization radius equal to s . It remains to show that ξ_k satisfies non-degeneracy (1.11). With s fixed, put $r := (3s)^d$. Let E_1 be the event that $\mathcal{P}_\lambda^{\beta\Psi}$ puts $k+1$ points in Q_{s^d} and no points in $Q_r \setminus Q_{s^d}$. On the event E_1 we have $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_k^{(s)}(x, \mathcal{P}_\lambda^{\beta\Psi}) = 1$. On the other hand, if E_2 is the event that $\mathcal{P}_\lambda^{\beta\Psi}$ puts fewer than $k+1$ points in Q_{s^d} and no points in $Q_r \setminus Q_{s^d}$ then $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_k^{(s)}(x, \mathcal{P}_\lambda^{\beta\Psi}) = 0$. Events E_1 and E_2 have strictly positive probability and give rise to different values of $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_k^{(s)}(x, \mathcal{P}_\lambda^{\beta\Psi})$, regardless of the point configurations $\mathcal{P}_\lambda^{\beta\Psi} \cap Q_r^c$. This shows (1.11) and concludes the proof. \square

2.2 Functionals of Euclidean graphs

Many functionals of Euclidean graphs on Gibbsian input satisfy (1.17) and (1.18), as shown in [35]. However [35] left open the question of showing variance lower bounds, which is essential to showing that (1.18) is meaningful. We now redress this and assert that the functionals in [35] satisfy non-degeneracy (1.11), and thus $\sigma^2(\xi, \tau) > 0$. We illustrate this for select functionals in [35], leaving it to the reader to verify this assertion for the remaining functionals, namely those arising in random sequential adsorption, component counts in random geometric graphs, and Gibbsian loss networks.

(i) k -nearest neighbors graph. The k -nearest neighbors (undirected) graph on the vertex set \mathcal{X} , denoted $NG(\mathcal{X})$, is the graph obtained by including $\{x, y\}$ as an edge whenever y is one of the k points nearest to x and/or x is one of the k points nearest to y . The k -nearest neighbors (directed) graph on \mathcal{X} , denoted $NG'(\mathcal{X})$, is obtained by placing a directed edge between each point and its k -nearest neighbors. In case $\mathcal{X} = \{x\}$ is a singleton, x has no nearest neighbor and the *nearest neighbor* distance for x is set by convention to 0.

Total edge length of k -nearest neighbors graph. Given $x \in \mathbb{R}^d$ and a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, the *nearest neighbors length functional* $\xi_{NG}(x, \mathcal{X})$ is one half the sum of the edge lengths of edges in $NG(\mathcal{X} \cup \{x\})$ which are incident to x . The total edge length of $NG(\mathcal{P}^{\beta\Psi} \cap Q_\lambda)$ is given by

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_{NG}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}).$$

Theorem 5.2 in [35] shows that W_λ satisfies the rate of convergence to the normal at (1.18). This follows since ξ_{NG} is translation invariant, exponentially stabilizing in the wide sense, and satisfies the moment condition (1.6) for all $q \in (2, \infty)$. However that theorem leaves open the question of variance lower bounds for $\text{Var}W_\lambda$ and thus the rate of convergence is possibly useless. The next result resolves this question and also gives a slightly better bound than that in [35].

Theorem 2.2 *We have $\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var}W_\lambda = \tau \sigma^2(\xi_{NG}, \tau) > 0$ and*

$$d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var}W_\lambda}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Proof. We need only show that non-degeneracy (1.11) holds and then apply Theorem 1.1 and (1.19). We do this by modifying the proof of Lemma 6.3 of [29]. This goes as follows. Let $C_0 := Q_1$, the unit cube centered at the origin. The annulus $Q_{4^d} \setminus C_0$ will be called the moat; notice that Q_{4^d} has edge length 4. Partition the annulus $Q_{6^d} \setminus Q_{4^d}$ into a finite collection \mathcal{U} of unit cubes. Now define the following events. Let E_2 be the event that there are no points in $\mathcal{P}_\lambda^{\beta\Psi}$ in the moat and there are at least $k+1$ points in each of the unit subcubes in \mathcal{U} . Let E_1 be the intersection of E_2 and the event that there is 1 point in C_0 ; let E_0 be the intersection of E_2 and the event that there are no points in C_0 . Then E_0 and E_1 have strictly positive probability. Put $Q_r := Q_{6^d}$, i.e., put $r = 6^d$.

Given any configuration $\mathcal{P}^{\beta\Psi} \cap Q_r^c$, then conditional on the event that E_0 occurs, the sum

$$\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_{NG}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$$

is strictly less than the same sum, conditional on the event E_1 . This is because on the event E_1 there are k additional edges crossing the moat, each of length at least 3.

Thus E_0 and E_1 are events with strictly positive probability which give rise to values of $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_{NG}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$ which differ by at least $3k$, a fixed amount. This demonstrates non-degeneracy (1.11). \square

(ii) Gibbs-Voronoi tessellations. Given $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, the set of points in \mathbb{R}^d closer to x than to any other point of \mathcal{X} is the interior of a possibly unbounded convex

polyhedral cell $C(x, \mathcal{X})$. The Voronoi tessellation induced by \mathcal{X} is the collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$. When \mathcal{X} is the realization of the Poisson point set \mathcal{P}_τ , this generates the Poisson-Voronoi tessellation of \mathbb{R}^d . Here, given the Gibbs point process $\mathcal{P}^{\beta\Psi}$, consider the Voronoi tessellation of this process, sometimes called the Ord process [24].

Total edge length of Gibbs-Voronoi tessellations. Given $\mathcal{X} \subset \mathbb{R}^2$, let $\xi_{\text{vor}}(x, \mathcal{X})$ denote one half the total edge length of the *finite* length edges in the cell $C(x, \mathcal{X} \cup \{x\})$ (thus we do not take infinite edges into account). The total edge length of the Voronoi graph on $\mathcal{P}^{\beta\Psi}$ is given by

$$W_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi_{\text{vor}}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}).$$

It may be shown [35] that ξ_{vor} is exponentially stabilizing in the wide sense (1.9), that it satisfies the moment condition (1.6) for $q \in (2, \infty)$, and, as in Theorem 5.4 of [35] that W_λ satisfies the rate of convergence to the normal as in (1.18).

However that theorem leaves open the question of variance lower bounds for $\text{Var}W_\lambda$ and thus the rate of convergence is possibly useless. The next result resolves this question and gives a better rate than that in [35].

Theorem 2.3 *We have $\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var}W_\lambda = \tau \sigma^2(\xi_{\text{vor}}, \tau) > 0$ and*

$$d_K \left(\frac{W_\lambda - \mathbb{E}W_\lambda}{\sqrt{\text{Var}W_\lambda}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Proof. We need only show that non-degeneracy (1.11) is satisfied and then apply Theorem 1.1 and (1.19). We do this by modifying the proof of Lemma 8.2 of [29]. This goes as follows.

Consider the construction used in the proof of Theorem 2.2. Let E_2 be the event that there are no points of $\mathcal{P}_\lambda^{\beta\Psi}$ in the moat and there is at least one point in each of the subcubes in \mathcal{U} . Fix ε small ($< 1/100$). Choose points $x_1, x_2, x_3 \in \mathbb{R}^2$ forming an equilateral triangle of side-length $1/2$, centered at the origin. Let A_0 be the intersection of E_2 and the event that there is exactly one point in each of $B_\varepsilon(x_i)$, and the event that there is no other point in $C_0 \setminus (\cup_{i=1}^3 B_\varepsilon(x_i))$, except for a point z in the ball $B_{\varepsilon\delta}(0)$, where $\delta \in (0, 1)$ will be chosen shortly. Let A_1 be the intersection of E_2 , the event that there is exactly one point in each of $B_{\varepsilon\delta}(\delta x_i)$, and the event that there is no other point in $C_0 \setminus (\cup_{i=1}^3 B_{\varepsilon\delta}(\delta x_i))$, except for the point z in the ball $B_{\varepsilon\delta}(0)$.

On the event A_0 , the presence of z near the origin leads to three edges, namely the edges of a (nearly equilateral) triangular cell T around the origin. It removes the parts of the three edges of the Voronoi graph (on all points except z) which intersect T . The difference between the sum of the lengths of the added edges and the sum of the lengths of the three removed edges exceeds some fixed positive number α (the reason is this:

given an equilateral triangle T , and a point P inside it, the sum of the lengths of the three edges joining P to the vertices of T is strictly less than the perimeter of T since the length of each of the three edges is less than the common length of the side of T . If T is nearly equilateral (our case) this is still true).

On the other hand, on the event A_1 , the presence of z cannot increase the total edge length by more than the total edge length of triangular cell around the origin, and this increase is bounded by a constant multiple of δ , which is less than α if δ is small enough. Thus if δ is small enough, the events A_0 and A_1 give rise to values of $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap Q_r} \xi_{\text{Vor}}(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\})$ which differ by at least some fixed amount. This demonstrates non-degeneracy (1.11). \square

2.3 Insurance models

The modeling of insurance claims has been of considerable interest in the literature. The thrust of the modeling is to set up a claim process $\{N_t, t \geq 0\}$ to record the number and time of claims and a sequence of random variables $\{X_i, i \geq 1\}$ representing the claim sizes. The aggregate claim size by time t can then be represented as $S_t = \sum_{i=1}^{N_t} X_i$. Most of the literature assumes that $\{X_i, i \geq 1\}$ are independent and identically distributed random variables, and are independent of the claim process $\{N_t, t \geq 0\}$ [15]. When $\{N_t, t \geq 0\}$ is a Poisson process, the process $\{S_t, t \geq 0\}$ becomes a compound Poisson process and is also known as the Cramér–Lundberg model ([15], p. 22). Significant effort has been devoted to generalize the model so that it represents real situations more closely, e.g., making the claim process a more general counting process such as a renewal process, a negative binomial process, or a stationary point process [34]. To address the interdependence of claim sizes, [4] introduces a strictly stationary process $\{Y_t, t \geq 0\}$ representing a random environment of the claims and a simple point process H on $[0, T] \times \mathbb{N}$ recording the times and sizes of clusters of claims. The total claim amount X_a for $a = (t, n)$ is assumed to be the sum of n independent and identically distributed random variables with distribution determined by the value of Y_t . Assuming that $\{Y_t\}$ is independent of H and both $\{Y_t\}$ and H are locally dependent with a ‘uniform dependence radius h_0 ’ such that for all $0 < t_1 < t_2 < \infty$, $Y|_{[t_1, t_2]}$ is independent of $Y|_{\mathbb{R}^+ \setminus (t_1 - h_0, t_2 + h_0)}$ and $H|_{[t_1, t_2] \times \mathbb{N}}$ is independent of $H|_{(\mathbb{R}^+ \setminus (t_1 - h_0, t_2 + h_0)) \times \mathbb{N}}$, [4] proves that the aggregate claim size $W_T := \int_{a=(t,n): t \leq T} X_a H(da)$, when standardized, can be approximated in distribution by the standard normal with an approximation error of order $O(T^{-1/2})$.

In disastrous events, insurance claims may involve dependence amongst the time, size and environment of the claims. In applications, local dependence with a uniform dependence radius may be violated. In this subsection, we aim to address these issues.

To this end, let the time and spatial location of claims of insurances be represented by $\mathcal{P}^{\beta\Psi}$, a Gibbs point process in $\mathbb{R}^+ \times \mathbb{R}^d$. In practice, we have $d \in \{2, 3\}$ and the space is typically restricted to a compact convex set $\mathbb{D} \subset \mathbb{R}^d$ with $\text{Vol}_d(\mathbb{D}) > 0$. Consequently, we set $\tilde{\mathcal{P}}_T^{\beta\Psi} := \mathcal{P}^{\beta\Psi}|_{[0, T] \times \mathbb{D}}$. Let $\xi((t, \mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi})$ be the value of the claim at (t, \mathbf{s}) with $t \in \mathbb{R}^+$ and $\mathbf{s} \in \mathbb{R}^d$. The aggregate claim size in the time interval $[0, T]$ is $\tilde{W}_T := \int_{[0, T] \times \mathbb{D}} \xi((t, \mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \tilde{\mathcal{P}}_T^{\beta\Psi}(dt, d\mathbf{s})$. The proof of the next result makes use of Lemma 4.2 and is thus deferred to Section 4.

Theorem 2.4 *Assume that ξ is exponentially stabilizing in the wide sense (1.9), translation invariant in the time coordinate t , and satisfies the q -moment condition (1.20) for some $q \in (3, \infty)$. If there exists an $\epsilon > 0$ such that for all large T there is an interval $I \subset (\epsilon T, (1 - \epsilon)T)$ of length $\Theta(1)$, such that the conditional distribution $\tilde{W}_T | \tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, then*

$$d_K \left(\frac{\tilde{W}_T - \mathbb{E} \tilde{W}_T}{\sqrt{\text{Var} \tilde{W}_T}}, N(0, 1) \right) = O((\ln T)^{3.5} T^{-1/2}).$$

Corollary 2.1 *Assume that the distribution of $\xi((t, \mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi})$ is determined by the k -nearest neighbors of (t, \mathbf{s}) and satisfies the q -moment condition (1.20) for some $q \in (3, \infty)$. If there exists an $\epsilon > 0$ such that for all large T there is an interval $I \subset (\epsilon T, (1 - \epsilon)T)$ of length $\Theta(1)$, such that the conditional distribution $\tilde{W}_T | \tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, then*

$$d_K \left(\frac{\tilde{W}_T - \mathbb{E} \tilde{W}_T}{\sqrt{\text{Var} \tilde{W}_T}}, N(0, 1) \right) = O((\ln T)^{3.5} T^{-1/2}).$$

Proof. Using the argument of Section 2.2 (i), one can easily verify that ξ satisfies all the conditions of Theorem 2.4, hence the conclusion follows. \square

2.4 Maximal points of Gibbsian samples

Let $K := [0, \infty)^d$. Given $\mathcal{X} \subset \mathbb{R}^d$ locally finite, $x \in \mathcal{X}$ is called K -maximal, or simply maximal if $(K \oplus x) \cap \mathcal{X} = \{x\}$. A point $x = (x_1, \dots, x_d) \in \mathcal{X}$ is maximal if there is no other point $(z_1, \dots, z_d) \in \mathcal{X}$ with $z_i \geq x_i$ for all $1 \leq i \leq d$. The maximal layer $m_K(\mathcal{X})$ is the collection of maximal points in \mathcal{X} . Let $M_K(\mathcal{X}) := \text{card}(m_K(\mathcal{X}))$.

Consider the region

$$A := \{(v, w) : v \in D, 0 \leq w \leq F(v)\}$$

where $F : D \rightarrow \mathbb{R}$ has continuous partials $F_i, 1 \leq i \leq d-1$, bounded away from zero and negative infinity, $D \subset [0, 1]^{d-1}$, and $|F| \leq 1$. We are interested in showing asymptotic

normality for $M_K([\lambda^{-1/d}\mathcal{P}_\lambda^{\beta\Psi} \oplus (1/2, \dots, 1/2)] \cap A)$, with $\mathcal{P}_\lambda^{\beta\Psi}$ as in (1.4). Maximal points are invariant with respect to scaling and translations and it suffices to prove a central limit theorem for $M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A)$.

The asymptotic behavior and central limit theorem for $M_K(\mathcal{X})$ with \mathcal{X} either Poisson or binomial input has been studied in [13, 1, 2, 3, 4]; the next theorem extends these results to Gibbsian input.

Theorem 2.5 *We have*

$$d_K \left(\frac{M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A) - \mathbb{E} M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A)}{\sqrt{\text{Var} M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A)}}, N(0, 1) \right) = O((\ln \lambda)^{(7d-1)/2} \lambda^{-(d-1)/2d}).$$

Proof. We shall show this is a consequence of Theorem 1.3 for an appropriate \tilde{S}_λ . For any subset $E \subset \mathbb{R}^d$ and $\epsilon > 0$ let $E^\epsilon := \{x \in \mathbb{R}^d : d(x, E) < \epsilon\}$, where $d(x, E)$ denotes the Euclidean distance between x and the set E . Put $\partial A := \{(v, F(v)) : v \in D\}$, $\tilde{S}_\lambda := (\lambda^{1/d}\partial A)^{c \ln \lambda}$ and in accordance with (1.12), we set $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_\lambda$. Given any $L \in [1, \infty)$ we observe that if c is large then $M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A) = M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d}A)$ with probability at least $1 - \lambda^{-L}$. Since the third moment of $M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A)$ is bounded by $O(\lambda^3)$, this is enough to guarantee that $\text{Var} M_K(\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A)$ and $\text{Var} M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d}A)$ have the same asymptotic behavior and thus it is enough to prove Theorem 2.5 with $\mathcal{P}^{\beta\Psi} \cap \lambda^{1/d}A$ replaced by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d}A$. Put

$$\zeta(x, \mathcal{X}) := \zeta(x, \mathcal{X}; \lambda^{1/d}A) := \begin{cases} 1 & \text{if } ((K \oplus x) \cap \lambda^{1/d}A) \cap (\mathcal{X} \cup \{x\}) = \{x\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that ζ is not translation invariant and that

$$M_K(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d}A) = \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \lambda^{1/d}A} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}).$$

To prove Theorem 2.5, it suffices to show that ζ satisfies exponential stabilization in the wide sense (1.9) and apply Theorem 1.3.

To show exponential stabilization, we argue as follows. Given $x \in \tilde{S}_\lambda \cap \lambda^{1/d}A$, let $D_1(x) := D_1(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$ be the distance between x and the nearest point in $(K \oplus x) \cap \lambda^{1/d}A \cap \tilde{\mathcal{P}}_\lambda^{\beta\Psi}$, if there is such a point; otherwise we let $D_1(x)$ be the *maximal* distance between x and $(K \oplus x) \cap \lambda^{1/d}\partial A$, denoted here by $D(x)$. By the smoothness assumptions on ∂A , it follows that $(K \oplus x) \cap \lambda^{1/d}A \cap B_t(x)$ has volume at least $c_1 t^d$ for all $t \in [0, D(x)]$. It follows that uniformly in $x \in \tilde{S}_\lambda \cap \lambda^{1/d}A$ and $\lambda \in [1, \infty)$

$$\mathbb{P}[D_1(x) > t] \leq \exp(-c_1 t^d), \quad 0 \leq t \leq D(x). \quad (2.1)$$

For $t \in (D(x), \infty)$, this inequality holds trivially and so (2.1) holds for all $t \in (0, \infty)$.

Let $R(x) := R(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi}) := D_1(x)$. We claim that $R := R(x)$ is a radius of stabilization for ζ at x . Indeed, if $D_1(x) \in (0, D(x))$, then x is not maximal, and so

$$\zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)) = 0$$

and inserting points \mathcal{Y} outside $B_R(x)$ does not modify the score ζ . If $D_1(x) \in [D(x), \infty)$ then

$$\zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)) = 1.$$

Keeping the realization $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap B_R(x)$ fixed, we notice that inserting points \mathcal{Y} outside $B_R(x)$ does not modify the score ζ , since maximality of x is preserved. Thus $R(x)$ is a radius of stabilization for ζ at x and it decays exponentially fast, as demonstrated above.

Clearly the moment condition (1.20) is satisfied since ζ is bounded by one. We now show that ζ satisfies non-degeneracy (1.13) for a large number of cubes of volume at least $c_2 r$. We do this for $d = 2$, but the proof extends to higher dimensions.

Fix $r \in [1, \infty)$. Let $\tilde{Q}_r \subset \tilde{S}_\lambda$ be such that $\tilde{Q}_r \cap \lambda^{1/d} \partial A \neq \emptyset$. We also assume that $\lambda^{1/d} A$ contains only the lower left corner of \tilde{Q}_r , but that $\text{Vol}(\tilde{Q}_r \cap \lambda^{1/d} A) \geq c_3 r$.

Referring to Figure 1, we consider the event E that $\text{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_1) = \text{card}(\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_2) = 1$, where S_1 and S_2 are the squares in Figure 1. Let E_1 be the event that $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no points in $\tilde{Q}_r \setminus (S_1 \cup S_2)$. Note that $\mathbb{P}[E \cap E_1]$ is bounded away from zero, uniformly in λ . On $E \cap E_1$ we have that

$$\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$$

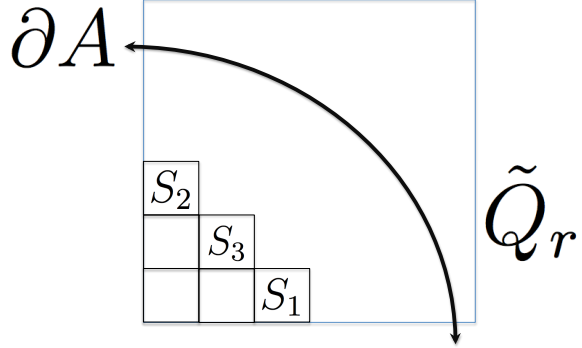


Figure 1: The square \tilde{Q}_r and the subsquares S_1, S_2, S_3

contributes a value of 2 to the total sum $\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$. Let E_2 be the event that $\tilde{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no points in $\tilde{Q}_r \setminus (S_1 \cup S_2)$, except for a singleton in the square S_3 . Then $\mathbb{P}[E \cap E_2]$ is bounded away from zero, uniformly in λ . On $E \cap E_2$ we have that

$$\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$$

contributes a value of 3 to the total sum $\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \zeta(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi})$. This is true regardless of the configuration $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_r^c$ and so condition (1.13) holds. Since the surface area of $\lambda^{1/d} \partial A$ is

$\Theta(\lambda^{(d-1)/d})$, the number of cubes \tilde{Q}_r having these properties is of order $\Theta((\lambda^{1/d}/\ln \lambda)^{d-1})$, whenever $\rho = \Theta(\ln \lambda)$. Thus we have $n(\rho, r, \tilde{S}_\lambda) = \Theta((\lambda^{1/d}/\ln \lambda)^{d-1})$.

Applying Theorem 1.3 we obtain Theorem 2.5. Noting that $\text{Vol}_d(\tilde{S}_\lambda) = \Theta(\lambda^{(d-1)/d} \ln \lambda)$, the bound (1.23) yields the rate of convergence to the normal

$$= O((\ln \lambda)^{2d} \lambda^{(d-1)/d} \ln \lambda (\lambda^{(d-1)/d}/(\ln \lambda)^{d-1})^{-3/2}) = O((\ln \lambda)^{7d/2-1/2} \lambda^{-(d-1)/2d}),$$

which was to be shown. \square

2.5 Spatial birth-growth models

Consider the following spatial birth-growth model on \mathbb{R}^d . Seeds appear at random locations $X_i \in \mathbb{R}^d$ at i.i.d. times T_i , $i = 1, 2, \dots$ according to a spatial-temporal point process $\mathcal{P} := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \infty)\}$. When a seed is born, it has initial radius zero and then forms a cell within \mathbb{R}^d by growing radially in all directions with a constant speed $v > 0$. Whenever one growing cell touches another, it stops growing in that direction. If a seed appears at X_i and if X_i belongs to any of the cells existing at the time T_i , then the seed is discarded. We assume that the law of $X_i, i \geq 1$, is independent of the law of $T_i, i \geq 1$.

Such growth models have received considerable attention with mathematical contributions given in [10, 11, 12, 17, 25]. First and second order characteristics for Johnson-Mehl growth models on homogeneous Poisson points on \mathbb{R}^d are given in [22, 23]. Using the general Theorem 1.2, we may extend many of these results to growth models with Gibbsian input. We illustrate with the following theorem, in which \mathcal{P} denotes a marked Gibbs point process with intensity measure $m^{\beta\psi} \times \mu$, where $m^{\beta\psi}$ is the intensity measure of $\mathcal{P}^{\beta\psi}$ and μ is an arbitrary probability measure on $[0, \infty)$.

Given a compact subset K' of \mathbb{R}^d , let $N(\mathcal{P}; K')$ be the number of seeds accepted in K' . We shall deduce the following result from Remark (iii) following Theorem 1.2. We let $\hat{\mathcal{P}}_\lambda^{\beta\psi}$ denote the process of marked points $\{(X_i, T_i) : X_i \in \mathcal{P}_\lambda^{\beta\psi}, T_i \in [0, \infty)\}$. Given a marked point set $\mathcal{X} \subset \mathbb{R}^d \times [0, \infty)$, define the score

$$\nu(x, \mathcal{X}) := \begin{cases} 1 & \text{if the seed at } x \text{ is accepted,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.6 *We have $\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var} N(\hat{\mathcal{P}}_\lambda^{\beta\psi}; Q_\lambda) = \tau \sigma^2(\nu, \tau) > 0$ and*

$$d_K \left(\frac{N(\hat{\mathcal{P}}_\lambda^{\beta\psi}; Q_\lambda) - \mathbb{E} N(\hat{\mathcal{P}}_\lambda^{\beta\psi}; Q_\lambda)}{\sqrt{\text{Var} N(\hat{\mathcal{P}}_\lambda^{\beta\psi}; Q_\lambda)}}, N(0, 1) \right) = O((\ln \lambda)^{2d} \lambda^{-1/2}).$$

Proof. Notice by the definition of ν we have

$$N(\hat{\mathcal{P}}_\lambda^{\beta\psi}; Q_\lambda) = \sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\psi} \cap Q_\lambda} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\psi}).$$

Let K denote the downward right circular cone with apex at the origin of \mathbb{R}^d . Then

$$\nu(x, \mathcal{X}) = \begin{cases} 1 & \text{if } (K \oplus x) \cap (\mathcal{X} \cup \{x\}) = x, \\ 0 & \text{otherwise.} \end{cases}$$

We now aim to show that ν satisfies all the conditions of Theorem 1.2. Clearly ν is translation invariant in \mathbb{R}^d . The moment condition (1.6) is satisfied, since $|\nu| \leq 1$. We claim that ν satisfies exponential stabilization in the wide sense. This however follows from the above proof that ζ is exponentially stabilizing in the wide sense (the proof is easier now because the boundary of A corresponds to the hyperplane \mathbb{R}^d).

We claim that non-degeneracy (1.11) holds. But this too follows from simple modifications of the proof of non-degeneracy of ζ . In fact things are easier, because we need only show that (1.11) holds for one cube Q_r . To this end, the cube Q_r is now replaced by a space-time cylinder $C_r := [-r^{1/d}, r^{1/d}]^d \times [0, \infty)$. For simplicity of exposition only, we show non-degeneracy for $d = 1$, but the approach extends to all dimensions.

Referring to Figure 2, we consider the event E that $\text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_1) = \text{card}(\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap S_2) = 1$. Let E_1 be the event that $\hat{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no other points in $([-r, r] \times [0, 1]) \setminus (S_1 \cup S_2)$ (we don't care about the point configuration in the set $[-r, r] \times [1, \infty)$). Note that $\mathbb{P}[E \cap E_1]$ is bounded away from zero, uniformly in λ . On $E \cap E_1$ we have that

$$\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_r} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$$

contributes a value of 2 to the total

sum $\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_t} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$. Let E_2 be the event that $\hat{\mathcal{P}}_\lambda^{\beta\Psi}$ puts no other points in $([-r, r] \times [0, 1]) \setminus (S_1 \cup S_2)$, except for a singleton in the diamond S_3 . Then $\mathbb{P}[E \cap E_2]$ is bounded away from zero, uniformly in λ . On $E \cap E_2$ we have that

$$\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_r} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$$

contributes a value of 3 to the total sum $\sum_{x \in \hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_t} \nu(x, \hat{\mathcal{P}}_\lambda^{\beta\Psi})$. This is true regardless of the configuration $\hat{\mathcal{P}}_\lambda^{\beta\Psi} \cap C_r^c$ and so condition (1.11) holds. Thus ν satisfies all conditions of Theorem 1.2 and so Theorem 2.6 follows. \square

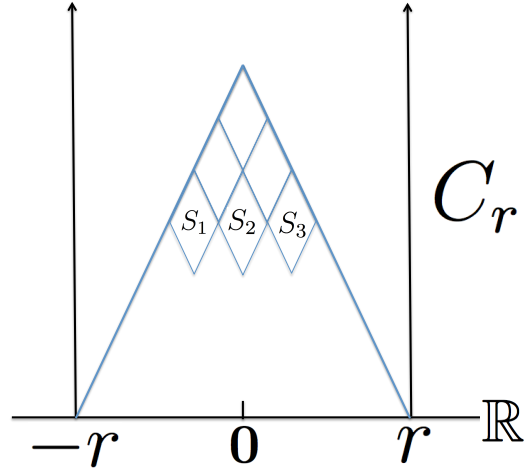


Figure 2: Space-time cylinder C_r

3 Auxiliary results

Before proving our main theorems we require a few additional results.

(i) **Control of spatial dependencies of Gibbs point processes.** Recall that $\mathcal{P}^{\beta\Psi}$ is an admissible point process, i.e., $\Psi \in \Psi^*$ and $(\tau, \beta) \in \mathcal{R}^\Psi$. As shown in the perfect simulation techniques of [35], the process has spatial dependencies which can be controlled by the size of the so-called *ancestor clans*. The ancestor clans are backwards in time oriented percolation clusters, where two nodes in space time are linked with a directed edge if one is the ancestor of the other. The acceptance status of a point at x depends on points in the ancestor clan. As seen at (3.6) of [35], the ancestor clans have exponentially decaying spatial diameter. Thus, if $A_B^{\beta\Psi}(t)$ is the ancestor clan in $\mathcal{P}^{\beta\Psi}$ of the set $B \subset \mathbb{R}^d$ at time t , then for all $(\tau, \beta) \in \mathcal{R}^\Psi$, there is a constant $c := c(\tau, \beta) \in (0, \infty)$ such that for all $t \in (0, \infty)$, $M \in (0, \infty)$, and $B \subset \mathbb{R}^d$ we have

$$\mathbb{P}[\text{diam}(A_B^{\beta\Psi}(t)) \geq M + \text{diam}(B)] \leq c(1 + \text{vol}(B)) \exp(-M/c). \quad (3.1)$$

Let $A_{B,\lambda}^{\beta\Psi}$ be the ancestor clan in $\mathcal{P}_\lambda^{\beta\Psi}$ of the set B . Since $\text{diam}(A_{B,\lambda}^{\beta\Psi}(t)) \leq \text{diam}(A_B^{\beta\Psi}(t))$, the bound (3.1) also holds for $A_{B,\lambda}^{\beta\Psi}$, i.e., for all $\lambda \in [1, \infty)$, $B \subset Q_\lambda$ we have

$$\mathbb{P}[\text{diam}(A_{B,\lambda}^{\beta\Psi}(t)) \geq M + \text{diam}(B)] \leq c(1 + \text{vol}(B)) \exp(-M/c).$$

Put for all $\rho \in (0, \infty)$

$$d(\rho) := \limsup_{\lambda \rightarrow \infty} \sup_{B \subset Q_\lambda, \text{diam}(B) \leq \rho/2} \mathbb{P}[\text{diam}(A_{B,\lambda}^{\beta\Psi}) \geq \rho].$$

Then we have

$$d(\rho) \leq c(1 + (\rho/2)^d v_d) \exp(-\rho/2c). \quad (3.2)$$

(ii) **Score functions with deterministic range of dependency.** Given the radius of stabilization $R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi})$, let $D(x, \mathcal{P}_\lambda^{\beta\Psi})$ be the diameter of the ancestor clan of the stabilization ball $B_{R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi})}(x)$. For all $\rho \in (0, \infty)$, consider score functions on points having ancestor clan diameter at most ρ :

$$\xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho) := \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \mathbf{1}(D(x, \mathcal{P}_\lambda^{\beta\Psi}) \leq \rho).$$

We study the following functional, the analog of $W(\rho)$ on page 704 of [4]:

$$W_\lambda(\rho) := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho). \quad (3.3)$$

When sets A and B are separated by a Euclidean distance greater than 2ρ , then the random variables $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap A} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho)$ and $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap B} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho)$ depend on disjoint and hence independent portions of the birth and death process $(\varrho(t))_{t \in \mathbb{R}}$ in the construction of $\mathcal{P}_\lambda^{\beta\Psi}$. We make heavy use of this in the proofs of Theorems 1.2 and 1.3.

It is also useful to consider sums of scores with respect to the global point process $\mathcal{P}^{\beta\Psi}$, namely

$$\hat{W}_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}); \quad \hat{W}_\lambda(\rho) := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho).$$

(iii) **Wide sense stabilization of ξ on $\mathcal{P}_\lambda^{\beta\Psi}$.** If ξ is a stabilizing functional in the wide sense, then

$$Q(\rho) := \limsup_{\lambda \rightarrow \infty} \sup_{x \in Q_\lambda} \mathbb{P}[R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi}) > \rho | \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\} = 1] \rightarrow 0,$$

as $\rho \rightarrow \infty$. If ξ is *exponentially stabilizing* in the wide sense (1.9), then by (1.10) there is a constant $c \in (0, \infty)$ such that

$$Q(\rho) \leq c \exp(-\rho/c). \quad (3.4)$$

Notice that for any $\rho \in (0, \infty)$ we have

$$\begin{aligned} & \mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho | \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\} = 1] \\ & \leq \mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho, R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi}) \leq \rho/2 | \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\} = 1] \\ & \quad + \mathbb{P}[R^\xi(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho/2 | \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\} = 1]. \end{aligned}$$

Bounding the first term on the right hand side by (3.2) and the second by (3.4), we obtain whenever $\rho \in [c' \ln \lambda, \infty)$ and c' is large that there is c_1 such that $\mathbb{P}[D(x, \mathcal{P}_\lambda^{\beta\Psi}) \geq \rho | \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\} = 1] \leq c_1 \exp(-\rho/c_1)$ whenever $\rho \in [c' \ln \lambda, \infty)$. Thus, for any $L \in [1, \infty)$, there is c large enough so that if $\rho \in [c \ln \lambda, \infty)$, then

$$\mathbb{P}[\hat{W}_\lambda \neq \hat{W}_\lambda(\rho)] \leq \lambda^{-L} \quad (3.5)$$

and

$$\mathbb{P}[W_\lambda \neq W_\lambda(\rho)] \leq \lambda^{-L}. \quad (3.6)$$

4 Variance and moment bounds

Let r satisfy non-degeneracy (1.11) and let $\rho \in [r, \infty)$. Find a maximal collection of disjoint cubes $Q_{i,r} := Q_{i,r,\rho} \subset Q_\lambda, i \in I$, with $\text{Vol}_d Q_{i,r} = r$, and which are separated

by a distance at least 4ρ and which are at least a distance 2ρ from ∂Q_λ . Notice that $n(\rho, Q_\lambda) := \text{card}(I) = \lfloor c'\lambda/\rho^d \rfloor$, c' a constant. Let \mathcal{F}_i be the sigma algebra generated by $\mathcal{P}^{\beta\Psi} \cap Q_{i,r}^c$. More precisely, letting \mathcal{B} be the class of all locally finite subsets of \mathbb{R}^d , define the sigma algebra \mathcal{B} in \mathcal{B} as the smallest sigma algebra making the mappings $\eta \in \mathcal{B} \mapsto \text{card}(\eta \cap \Theta)$, for all Borel sets $\Theta \subset \mathbb{R}^d$, measurable (see [18], page 12). The sigma algebra \mathcal{F}_i is induced by the mapping $\mathcal{P}^{\beta\Psi} \mapsto \mathcal{P}^{\beta\Psi} \cap Q_{i,r}^c$ from \mathcal{B} to $(\mathcal{B}, \mathcal{B})$.

Lemma 4.1 *Let $q \in [1, \infty)$. If ξ satisfies the moment condition (1.6) for some $q' \in (q, \infty)$ then there are constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \geq \lambda_0$ and $\rho \in [1, \infty)$*

$$\max\{\|W_\lambda\|_q, \|W_\lambda(\rho)\|_q\} \leq c\lambda \quad (4.1)$$

and

$$\sup_{i \in I} \max\{\|\mathbb{E}[W_\lambda | \mathcal{F}_i]\|_q, \|\mathbb{E}[W_\lambda(\rho) | \mathcal{F}_i]\|_q\} \leq c\lambda.$$

Identical bounds hold if W_λ is replaced by \hat{W}_λ .

Proof. Fix $q \in [1, \infty)$. We shall only prove $\|W_\lambda\|_q \leq c\lambda$ as the other inequalities follow similarly. Put $N := \text{card}(\mathcal{P}_\lambda^{\beta\Psi})$. Minkowski's inequality gives

$$\begin{aligned} \|W_\lambda\|_q &\leq \sum_{j=0}^{\infty} \left\| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \mathbf{1}(\lambda\tau 2^j \leq N \leq \lambda\tau 2^{j+1}) \right\|_q \\ &\leq \sum_{j=0}^{\infty} \left\| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}, N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \mathbf{1}(N \geq \lambda\tau 2^j) \right\|_q. \end{aligned}$$

Let $s \in (1, \infty)$ be such that $qs < q'$. Let $1/s + 1/t = 1$, i.e., s and t are conjugate exponents. Hölder's inequality gives

$$\|W_\lambda\|_q \leq \sum_{j=0}^{\infty} \left[\mathbb{E} \left(\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}, N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \right)^{qs} \right]^{1/qs} (\mathbb{P}[N \geq \lambda\tau 2^j])^{1/qt}.$$

Since $\mathcal{P}_\lambda^{\beta\Psi}$ is Poisson-like, we have that N is stochastically dominated by a Poisson random variable $\text{Po}(\lambda\tau)$ with parameter $\lambda\tau$. Recalling the definition of w_q at (1.6), we obtain

$$\begin{aligned} \|W_\lambda\|_q &\leq \sum_{j=0}^{\infty} \left\| \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}, N \leq \lambda\tau 2^{j+1}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}) \right\|_{qs} (\mathbb{P}[\text{Po}(\lambda\tau) \geq \lambda\tau 2^j])^{1/qt} \\ &\leq 6\lambda\tau w_{qs} + \sum_{j=2}^{\infty} \lambda\tau 2^{j+1} w_{qs} (\mathbb{P}[\text{Po}(\lambda\tau) - \lambda\tau \geq \lambda\tau(2^j - 1)])^{1/qt}, \end{aligned}$$

using Minkowski's inequality another time. For $j \geq 2$, we have that $\mathbb{P}[\text{Po}(\lambda\tau) - \lambda\tau \geq \lambda\tau(2^j - 1)]$ decays exponentially fast in 2^j by standard tail probabilities for the Poisson random variable. This shows that the infinite sum is $O(\lambda\tau)$, concluding the proof. \square

We put

$$\tilde{W}_\lambda(\rho) := \sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}; \rho).$$

Lemma 4.2 *Given a set $G \subset \mathbb{R}^d$ we let \mathcal{G}_G (respectively $\tilde{\mathcal{G}}_G$) be the sigma algebra generated by $\mathcal{P}^{\beta\Psi} \cap G$ (respectively $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap G$). Assume that ξ satisfies condition (1.9).*

(a) If ξ satisfies the moment condition (1.6) for some $q \in (2, \infty)$, then there exist constants λ_0 and c such that for all $\lambda \in [\lambda_0, \infty)$, $\rho \in [c \ln \lambda, \infty)$ and all Borel sets $G \subset \mathbb{R}^d$,

$$|\mathbb{E} \text{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E} \text{Var}[\tilde{W}_\lambda|\mathcal{G}_G]| \leq \lambda^{-1} \quad (4.2)$$

and

$$|\mathbb{E} \text{Var}[W_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E} \text{Var}[W_\lambda|\mathcal{G}_G]| \leq \lambda^{-1}. \quad (4.3)$$

(b) If ξ satisfies the moment condition (1.20) for some $q \in (2, \infty)$ then there exist constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \in [\lambda_0, \infty)$, $\rho \in [c \ln \lambda, \infty)$ and all Borel sets $G \subset \tilde{S}_\lambda$,

$$|\mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho)|\tilde{\mathcal{G}}_G] - \mathbb{E} \text{Var}[\tilde{W}_\lambda|\tilde{\mathcal{G}}_G]| \leq \lambda^{-1}.$$

Proof. (a) Using the generic formula $\text{Var}[X|\mathcal{A}] = \mathbb{E}[X^2|\mathcal{A}] - (\mathbb{E}[X|\mathcal{A}])^2$, valid for any random variable X and sigma algebra \mathcal{A} , we have

$$\mathbb{E} \text{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] = \mathbb{E} \left[\mathbb{E}[\hat{W}_\lambda^2(\rho)|\mathcal{G}_G] - (\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G])^2 \right]$$

and

$$\mathbb{E} \text{Var}[\hat{W}_\lambda|\mathcal{G}_G] = \mathbb{E} \left[\mathbb{E}[\hat{W}_\lambda^2|\mathcal{G}_G] - (\mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G])^2 \right].$$

If both differences

$$|\mathbb{E}[\mathbb{E}[\hat{W}_\lambda^2(\rho)|\mathcal{G}_G] - \mathbb{E}[\hat{W}_\lambda^2|\mathcal{G}_G]]| \quad (4.4)$$

and

$$|\mathbb{E}[\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G]^2 - \mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G]^2]| \quad (4.5)$$

are less than $\lambda^{-1}/2$ then $\mathbb{E} \text{Var}[\hat{W}_\lambda(\rho)|\mathcal{G}_G]$ differs from $\mathbb{E} \text{Var}[\hat{W}_\lambda|\mathcal{G}_G]$ by less than λ^{-1} .

Notice that (4.4) may be bounded by $(2\lambda)^{-1}$ since it equals $\mathbb{E}[\hat{W}_\lambda^2(\rho) - \hat{W}_\lambda^2]$, which by Hölder's inequality is bounded by the product of $\|\hat{W}_\lambda^2(\rho) - \hat{W}_\lambda^2\|_{q/2}$ and a power of $\mathbb{P}[\hat{W}_\lambda \neq \hat{W}_\lambda(\rho)]$. The first term is $O(\lambda^2)$ by (4.1) whereas the latter is small by (3.5), the choice of ρ , and the arbitrariness of L .

Likewise (4.5) can be bounded by $\lambda^{-1}/2$ since

$$\begin{aligned}
& |\mathbb{E}[\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G]^2 - \mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G]^2]| \\
&= |\mathbb{E}(\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] + \mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G])(\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G])| \\
&\leq C\lambda\|\mathbb{E}[\hat{W}_\lambda(\rho)|\mathcal{G}_G] - \mathbb{E}[\hat{W}_\lambda|\mathcal{G}_G]\|_2 \\
&\leq C\lambda\sqrt{\mathbb{E}\left(\mathbb{E}\left((\hat{W}_\lambda(\rho) - \hat{W}_\lambda)^2|\mathcal{G}_G\right)\right)} = C\lambda\sqrt{\mathbb{E}(\hat{W}_\lambda(\rho) - \hat{W}_\lambda)^2},
\end{aligned}$$

where the first inequality follows by the Cauchy-Schwarz inequality and Lemma 4.1 and where the second inequality follows by the conditional Jensen inequality. Using Hölder's inequality and the bound (3.5), we get that (4.5) is bounded by $\lambda^{-1}/2$, concluding the proof of (4.2). The proofs of (4.3) and part (b) follow the proof of (a) verbatim. \square

Proof of Theorem 2.4. We take $\tilde{S}_T := [0, T] \times \mathbb{D}$ in Theorem 1.3 and let r be the length of I . Let $n(\rho, r, \tilde{S}_T)$ be the maximum number of subsets $S_i \subset \tilde{S}_T$ of the form $(I + t_i) \times \mathbb{D}$, $t_i \in \mathbb{R}^+$, in \tilde{S}_T which are separated by 4ρ with $\rho = \Theta(\ln T)$. Then $\text{Vol}_{d+1}(\tilde{S}_T) = \Theta(T)$ and $n(\rho, r, \tilde{S}_T) = \Theta(T(\ln T)^{-1})$. Let $\tilde{\mathcal{P}}_T^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap \tilde{S}_T$ in accordance with (1.12). We show that (1.13) is satisfied for all S_i , $1 \leq i \leq n(\rho, r, \tilde{S}_T)$ and then apply Theorem 1.3 to $\tilde{\mathcal{P}}_T^{\beta\Psi}$. Since the conditional distribution $\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}$ is non-degenerate, we have

$$\mathbb{E} \text{Var}[\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}] := d_0 > 0.$$

For $J \subset [0, T] \times \mathbb{D}$, we define

$$M(J) := \int_J \xi((t, \mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \mathbf{1}(D((t, \mathbf{s}), \tilde{\mathcal{P}}_T^{\beta\Psi}) \leq \rho) \tilde{\mathcal{P}}_T^{\beta\Psi}(dt, ds).$$

Then

$$\begin{aligned}
& \mathbb{E} \text{Var}[M(\tilde{S}_T)|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \setminus S_i\}] \\
&= \mathbb{E} \text{Var}[M(S_i^{2\rho})|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \setminus S_i\}] \\
&= \mathbb{E} \text{Var}[M((I \times \mathbb{D})^{2\rho})|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}] \quad (\text{by translation invariance of } \xi) \\
&= \mathbb{E} \text{Var}[M(\tilde{S}_T)|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{([0, T] \setminus I) \times \mathbb{D}\}] \geq d_0 - O(T^{-1}),
\end{aligned}$$

where the inequality is due to Lemma 4.2(b). Using Lemma 4.2(b) again, we conclude that, for T large,

$$\mathbb{E} \text{Var}[\tilde{W}_T|\tilde{\mathcal{P}}_T^{\beta\Psi} \cap \{\tilde{S}_T \setminus S_i\}] \geq d_0 - O(T^{-1}).$$

All conditions of Theorem 1.3 are satisfied and it follows from (1.23) that

$$d_K \left(\frac{\tilde{W}_T - \mathbb{E} \tilde{W}_T}{\sqrt{\text{Var} \tilde{W}_T}}, N(0, 1) \right) = O((\ln T)^2 \text{Vol}(\tilde{S}_T) n(\rho, r, \tilde{S}_T)^{-3/2}) = O((\ln T)^{3.5} T^{-1/2}),$$

completing the proof. \square

Lemma 4.3 Assume that ξ is translation invariant and the moment condition (1.6) holds for some $q \in (2, \infty)$. Under conditions (1.9) and (1.11) there exist constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \in [\lambda_0, \infty)$ and all $\rho \in [c \ln \lambda, \infty)$ we have

$$\text{Var}[W_\lambda(\rho)] \geq c^{-1} b_0 \lambda \rho^{-d}; \quad \text{Var}[\hat{W}_\lambda(\rho)] \geq c^{-1} b_0 \lambda \rho^{-d}. \quad (4.6)$$

Proof. We only prove the first inequality as the second follows from identical methods. Let $c \geq 2/c'$ such that Lemma 4.2(a) holds, where c' is the constant such that the cardinality of I is $\lfloor c' \lambda / \rho^d \rfloor$. Let \mathcal{F} be the sigma algebra generated by $\mathcal{P}^{\beta\Psi} \cap (\bigcup_{i \in I} Q_{i,r})^c$. By the conditional variance formula

$$\text{Var}[W_\lambda(\rho)] = \text{Var}[\mathbb{E}[W_\lambda(\rho)|\mathcal{F}]] + \mathbb{E} \text{Var}[W_\lambda(\rho)|\mathcal{F}] \geq \mathbb{E} \text{Var}[W_\lambda(\rho)|\mathcal{F}].$$

Let $C_i := \{x \in \mathbb{R}^d : d(x, Q_{i,r}) \leq \rho\}$. Then the C_i are separated by 2ρ because the $Q_{i,r}$ are separated by at least 4ρ (this is the reason why we chose the 4ρ separation in the first place). Also, the C_i are contained in Q_λ .

For each $i \in I$ the sum $\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap C_i} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho)$ depends on points distant at most ρ from C_i . Thus the random variable $\mathbb{E}[W_\lambda(\rho)|\mathcal{F}]$ is a sum of independent random variables since the C_i are separated by 2ρ . Thus we obtain

$$\begin{aligned} \mathbb{E} \text{Var}[W_\lambda(\rho)|\mathcal{F}] &= \mathbb{E} \text{Var}\left[\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho) \middle| \mathcal{F}\right] \\ &= \mathbb{E} \sum_{i \in I} \text{Var}\left[\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap C_i} \xi(x, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{x\}; \rho) \middle| \mathcal{F}\right]. \end{aligned} \quad (4.7)$$

Recall that $E^\epsilon = \{x \in \mathbb{R}^d : d(x, E) < \epsilon\}$ for any set E and $\epsilon > 0$. For all $i \in I$, the restrictions of \mathcal{F} and \mathcal{F}_i to C_i^ρ coincide. For $x \in C_i$, we have that $\xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho)$ depends only on points in C_i^ρ and so we may thus replace \mathcal{F} with \mathcal{F}_i . Since $\mathcal{P}_\lambda^{\beta\Psi}$ and $\mathcal{P}^{\beta\Psi}$ coincide on C_i^ρ we may also replace $\xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho)$ with $\xi(x, \mathcal{P}^{\beta\Psi}; \rho)$. Also, we may replace the range of summation $x \in \mathcal{P}_\lambda^{\beta\Psi} \cap C_i$ by $x \in \mathcal{P}_\lambda^{\beta\Psi}$ because the conditional sum

$$\sum_{x \in \mathcal{P}^{\beta\Psi} \cap C_i^c \cap Q_\lambda} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}; \rho) | \mathcal{F}_i$$

is constant (indeed, if $x \in C_i^c$, then $\xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}; \rho)$ won't be affected by points in $Q_{i,r}$).

This yields

$$\mathbb{E} \text{Var}[W_\lambda(\rho)|\mathcal{F}] = \mathbb{E} \sum_{i \in I} \text{Var}\left[\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}; \rho) \middle| \mathcal{F}_i\right]. \quad (4.8)$$

By Lemma 4.2(a) for all $i \in I$,

$$\mathbb{E} \text{Var} \left[\sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}; \rho) | \mathcal{F}_i \right] \geq b_0/2.$$

Thus

$$\text{Var}[W_\lambda(\rho)] \geq \mathbb{E} \text{Var}[W_\lambda(\rho) | \mathcal{F}] \geq \mathbb{E} \sum_{i \in I} b_0/2 \geq b_0 c^{-1} \lambda \rho^{-d}.$$

□

Roughly speaking, the factor $\lambda \rho^{-d}$ in (4.6) is the cardinality of I , the index set of cubes of volume r , separated by 4ρ , and having the property that the total score on each cube has positive variability. For score functions which may not be translation invariant and/or are defined on a subset \tilde{S}_λ of \mathbb{R}^d , we have the following analog of Lemma 4.3. Recall the definition of $n(\rho, r, \tilde{S}_\lambda)$ right after (1.13).

Lemma 4.4 *Assume the moment condition (1.20) holds for some $q \in (2, \infty)$. Under conditions (1.9) and (1.13) there exist constants $\lambda_0 \in (0, \infty)$ and $c \in (0, \infty)$ such that for all $\lambda \in [\lambda_0, \infty)$ and all $\rho \in [c \ln \lambda, \infty)$ we have*

$$\text{Var}[\tilde{W}_\lambda(\rho)] \geq c^{-1} b_0 n(\rho, r, \tilde{S}_\lambda).$$

Proof. We follow the proof of Lemma 4.3. We write $\{\tilde{Q}_{i,r} : i \in \tilde{I}\} := \mathcal{C}(\rho, r, \tilde{S}_\lambda)$, the collection of cubes defined after (1.13). Let $\tilde{\mathcal{F}}_\lambda$ be the sigma algebra generated by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap (\bigcup_{i \in \tilde{I}} \tilde{Q}_{i,r})^c$. By the conditional variance formula

$$\text{Var}[\tilde{W}_\lambda(\rho)] = \text{Var}[\mathbb{E}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda]] + \mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda] \geq \mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda].$$

For $i \in \tilde{I}$, let $\tilde{C}_i := \{x \in \tilde{S}_\lambda : d(x, \tilde{Q}_{i,r}) \leq \rho\}$. Then the \tilde{C}_i are separated by 2ρ because the $\tilde{Q}_{i,r}$ are separated by at least 4ρ . Also, the \tilde{C}_i are contained in \tilde{S}_λ .

For each $i \in \tilde{I}$ the sum $\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{C}_i} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}; \rho)$ depends on points distant at most ρ from \tilde{C}_i . Thus $\mathbb{E}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda]$ is a sum of independent random variables since the \tilde{C}_i are separated by 2ρ . Thus we obtain the analog of (4.7), namely

$$\mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda] = \mathbb{E} \sum_{i \in \tilde{I}} \text{Var} \left[\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{C}_i} \xi(x, \tilde{\mathcal{P}}_\lambda^{\beta\Psi} \setminus \{x\}; \rho) | \tilde{\mathcal{F}}_\lambda \right].$$

Let $\tilde{\mathcal{F}}_{\lambda,i}$ be the sigma algebra generated by $\tilde{\mathcal{P}}_\lambda^{\beta\Psi} \cap \tilde{Q}_{i,r}$. For all $i \in \tilde{I}$, the restrictions of $\tilde{\mathcal{F}}_\lambda$ and $\tilde{\mathcal{F}}_{\lambda,i}$ to $\tilde{C}_i^\rho \cap \tilde{S}_\lambda$ coincide.

As in the proof of Lemma 4.3, we obtain the analog of (4.8), namely

$$\mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda] = \mathbb{E} \sum_{i \in \tilde{I}} \text{Var} \left[\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}^{\beta\Psi} \setminus \{x\}; \rho) | \tilde{\mathcal{F}}_{\lambda,i} \right].$$

If $\lambda \in [\lambda_0, \infty)$ and if λ_0 is large enough, then by Lemma 4.2(b) for all $i \in \tilde{I}$,

$$\mathbb{E} \text{Var} \left[\sum_{x \in \tilde{\mathcal{P}}_\lambda^{\beta\Psi}} \xi(x, \tilde{\mathcal{P}}^{\beta\Psi} \setminus \{x\}; \rho) | \tilde{\mathcal{F}}_{\lambda,i} \right] \geq b_0/2.$$

Thus

$$\text{Var}[\tilde{W}_\lambda(\rho)] \geq \mathbb{E} \text{Var}[\tilde{W}_\lambda(\rho) | \tilde{\mathcal{F}}_\lambda] \geq \mathbb{E} \sum_{i \in \tilde{I}} b_0/2 \geq b_0 \cdot \text{card}(\tilde{I}).$$

□

Lemma 4.5 *If the moment condition (1.6) holds for some $q \in (2, \infty)$ then $|\text{Var}W_\lambda - \text{Var}\hat{W}_\lambda| = o(\lambda)$.*

Proof. Put $\rho = c \ln \lambda$, c large. By (4.3) and (4.2) with $G = \emptyset$ we have $|\text{Var}W_\lambda(\rho) - \text{Var}W_\lambda| = o(1)$ and $|\text{Var}\hat{W}_\lambda(\rho) - \text{Var}\hat{W}_\lambda| = o(1)$. So it is enough to prove $|\text{Var}W_\lambda(\rho) - \text{Var}\hat{W}_\lambda(\rho)| = o(\lambda)$. We have

$$|\text{Var}W_\lambda(\rho) - \text{Var}\hat{W}_\lambda(\rho)| \leq \text{Var}(W_\lambda(\rho) - \hat{W}_\lambda(\rho)) + 2\text{cov}(W_\lambda(\rho) - \hat{W}_\lambda(\rho), \hat{W}_\lambda(\rho)).$$

The scores $\xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho)$ and $\xi(x, \mathcal{P}^{\beta\Psi}; \rho)$ coincide when $x \in Q_\lambda$ is distant at least ρ from ∂Q_λ . Thus $W_\lambda(\rho) - \hat{W}_\lambda(\rho) = U_\lambda - V_\lambda$, where

$$U_\lambda := \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi} \cap (\partial Q_\lambda)^\rho} \xi(x, \mathcal{P}_\lambda^{\beta\Psi}; \rho); \quad V_\lambda := \sum_{x \in \mathcal{P}^{\beta\Psi} \cap (\partial Q_\lambda)^\rho} \xi(x, \mathcal{P}^{\beta\Psi}; \rho).$$

Lemma 4.1 with $q = 2$ and $q' > 2$ ensures $\text{Var}U_\lambda$ and $\text{Var}V_\lambda$ are both of order $O((\text{Vol}(\partial Q_\lambda)^\rho)^2)$.

These bounds and the formula $\text{Var}[U_\lambda - V_\lambda] = \text{Var}U_\lambda + \text{Var}V_\lambda - 2\text{Cov}[U_\lambda, V_\lambda]$ shows that $\text{Var}[U_\lambda - V_\lambda] = o(\lambda)$. By the Cauchy-Schwarz inequality and Lemma 4.1, we obtain $\text{cov}[W_\lambda(\rho) - \hat{W}_\lambda(\rho), \hat{W}_\lambda(\rho)] = o(\lambda)$ as well. □

We need one more lemma. It shows that if fluctuations of \hat{W}_λ are not of volume order then they are necessarily at most of surface order and vice versa. A version of this dichotomy appears in the statistical physics literature [21] and also in [6]. We do not have any natural examples of \hat{W}_λ which are defined on all of Q_λ and which have fluctuations at most of surface order. However, when ancestor clans and stabilization radii have slowly decaying tails we expect that $\text{Var}\hat{W}_\lambda$ behaves less like a sum of i.i.d. random variables and more like a sum of random variables with very long range dependencies, presumably giving rise to smaller fluctuations. When the score at x is allowed to depend on nearby point configurations as well as on nearby scores, then Martin and Yalcin [21] establish conditions giving surface order fluctuations.

Lemma 4.6 *Let ξ be translation invariant. Either $\text{Var}\hat{W}_\lambda = \Omega(\lambda)$ or $\text{Var}\hat{W}_\lambda = O(\lambda^{(d-1)/d})$.*

Proof. Recall the definitions of $c^\xi(x)$ and $c^\xi(x, y)$ at (1.14) and (1.15), respectively. Similar to the proof of Theorem 2.2 of [35], by the integral characterization of Gibbs point processes, as in Chapter 6.4 of [24], it follows from the Georgii-Nguyen-Zessin formula that

$$\text{Var} \hat{W}_\lambda = \text{Var} \sum_{x \in \mathcal{P}_\lambda^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi} \setminus \{x\}) = \tau \int_{Q_\lambda} c^{\xi^2}(x) dx - \tau^2 \int_{Q_\lambda} \int_{Q_\lambda} c^\xi(x, y) dy dx.$$

Note that $c^\xi(x, y)$ decays exponentially fast with $|x - y|$, as shown in Lemmas 3.4 and 3.5 of [35]. By translation invariance of ξ and stationarity of $\mathcal{P}^{\beta\Psi}$ we get

$$\begin{aligned} \text{Var} \hat{W}_\lambda &= \tau c^{\xi^2}(\mathbf{0}) \lambda - \tau^2 \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y - x) \mathbf{1}(y \in Q_\lambda) dy dx \\ &= \tau c^{\xi^2}(\mathbf{0}) \lambda - \tau^2 \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \mathbf{1}(x + y \in Q_\lambda) dy dx := I_\lambda + II_\lambda. \end{aligned} \quad (4.9)$$

Now

$$\lambda^{-1} II_\lambda = -\tau^2 \lambda^{-1} \int_{Q_\lambda} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \mathbf{1}(x \in Q_\lambda - y) dy dx$$

and writing $\mathbf{1}(x \in Q_\lambda - y)$ as $1 - \mathbf{1}(x \in (Q_\lambda - y)^c)$ gives

$$\lambda^{-1} II_\lambda = -\tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy + \lambda^{-1} \tau^2 \int_{\mathbb{R}^d} \int_{Q_\lambda} c^\xi(\mathbf{0}, y) \mathbf{1}(x \in \mathbb{R}^d \setminus (Q_\lambda - y)) dx dy.$$

As in [21], for all $y \in \mathbb{R}^d$, put $\gamma_{Q_\lambda}(y) := \text{Vol}_d(Q_\lambda \cap (\mathbb{R}^d \setminus (Q_\lambda - y)))$. Then

$$\lambda^{-1} \text{Var} \hat{W}_\lambda = \lambda^{-1} I_\lambda + \lambda^{-1} II_\lambda = \tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy + \lambda^{-1} \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \gamma_{Q_\lambda}(y) dy. \quad (4.10)$$

Now we assert that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \gamma_{Q_\lambda}(y) dy = 0. \quad (4.11)$$

Indeed, by Lemma 1 of [21], we have $\lambda^{-1} \gamma_{Q_\lambda}(y) \rightarrow 0$ and since $\lambda^{-1} c^\xi(\mathbf{0}, y) \gamma_{Q_\lambda}(y)$ is dominated by $c^\xi(\mathbf{0}, y)$, which decays exponentially fast, the result follows by the dominated convergence theorem. Collecting terms in (4.9)-(4.11) and recalling (1.16) gives

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var} \hat{W}_\lambda = \tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy = \tau \sigma^2(\xi, \tau) \in [0, \infty), \quad (4.12)$$

where we note $\sigma^2(\xi, \tau)$ is finite by the exponential decay of $c^\xi(\mathbf{0}, y)$ as shown in Lemma 3.5 of [35].

It follows that if $\text{Var} \hat{W}_\lambda$ is not of volume order then we have $\tau c^{\xi^2}(\mathbf{0}) - \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) dy = 0$. Using this identity in (4.10), multiplying (4.10) by $\lambda^{1/d}$, and taking limits gives

$$\lim_{\lambda \rightarrow \infty} \lambda^{-(d-1)/d} \text{Var} \hat{W}_\lambda = \lim_{\lambda \rightarrow \infty} \tau^2 \lambda^{-(d-1)/d} \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \gamma_{Q_\lambda}(y) dy. \quad (4.13)$$

Now as in [21], we have $\lambda^{-(d-1)/d}\gamma_{Q_\lambda}(y) \leq C|y|$, showing that the integrand in (4.13) is dominated by an integrable function. By Lemma 1 of [21], there is a function $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-(d-1)/d} \gamma_{Q_\lambda}(y) = \gamma(y).$$

By dominated convergence we get the desired result:

$$\lim_{\lambda \rightarrow \infty} \lambda^{-(d-1)/d} \text{Var} \hat{W}_\lambda = \tau^2 \int_{\mathbb{R}^d} c^\xi(\mathbf{0}, y) \gamma(y) dy < \infty,$$

where once again the integral is finite by the exponential decay of $c^\xi(\mathbf{0}, y)$. \square

5 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. Combining (4.12) and Lemma 4.5 we obtain $\lim_{\lambda \rightarrow \infty} \lambda^{-1} \text{Var} W_\lambda = \tau \sigma^2(\xi, \tau)$, giving (1.17). Now assume non-degeneracy (1.11) and put $\rho = c \ln \lambda$. By Lemma 4.3 we have

$$\lim_{\lambda \rightarrow \infty} \lambda^{-(d-1)/d} \text{Var} \hat{W}_\lambda(\rho) = \infty$$

and therefore by (4.2) with $G = \emptyset$ we have $\lim_{\lambda \rightarrow \infty} \lambda^{-(d-1)/d} \text{Var} \hat{W}_\lambda = \infty$. By Lemma 4.6 we have $\text{Var} \hat{W}_\lambda = \Omega(\lambda)$ and Lemma 4.5 gives $\sigma^2(\xi, \tau) > 0$, as desired. \square

Proof of Theorem 1.2. We use a result based on the Stein method to derive rates of normal convergence. We follow the set-up of [4], as this yields rates which are a slight improvement over the methods of [35]. Given an admissible Gibbs point process $\mathcal{P}_\lambda^{\beta\Psi}$ with both β and Ψ fixed, we shall simply write \mathcal{P}_λ for $\mathcal{P}_\lambda^{\beta\Psi}$. Our first goal is to get rates of normal convergence for $W_\lambda(\rho)$ defined at (3.3). Then we use this to obtain rates for W_λ . Without loss of generality, we assume $p \in (2, q)$ and we show for all $\rho \in (0, \infty)$:

$$d_K \left(\frac{W_\lambda(\rho) - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var}(W_\lambda(\rho))}}, N(0, 1) \right) = O \left((\text{Var} W_\lambda(\rho))^{-p/2} \lambda w_q^p \rho^{d(p-1)} + (\text{Var} W_\lambda(\rho))^{-1/2} w_q \rho^d \right) \quad (5.1)$$

and, if (1.11) holds and if (1.6) holds for some $q \in (3, \infty)$,

$$d_K \left(\frac{W_\lambda(\rho) - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var}(W_\lambda(\rho))}}, N(0, 1) \right) = O \left(\rho^{2d} \lambda^{-1/2} \right). \quad (5.2)$$

The proof goes as follows. The local dependence condition LD3 of [4] requires for each $x \in Q_\lambda$ three nested neighborhoods A_x , B_x and C_x which satisfy $B_r(x) \subset A_x \subset B_x \subset C_x$ as $r \downarrow 0$ and such that the sum of scores over points in $B_r(x)$ (resp. A_x , B_x) are independent of the sum of scores over points in $(A_x^r)^c$ (resp. B_x^c , C_x^c). We claim that $W_\lambda(\rho)$ satisfies the local dependence condition LD3 with the neighborhoods

$A_x := B_{2\rho}(x)$, $B_x := B_{4\rho}(x)$ and $C_x := B_{6\rho}(x)$, $x \in Q_\lambda$. Indeed, this follows immediately since $\xi(\cdot, \mathcal{P}_\lambda^{\beta\Psi} \setminus \{\cdot\}; \rho)$ enjoys spatial independence over sets separated by more than 2ρ , as already noted in the discussion after (3.3).

It follows from Corollary 2.2 of [4] that

$$d_K \left(\frac{W_\lambda(\rho) - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var}(W_\lambda(\rho))}}, N(0, 1) \right) \leq 48\varepsilon_3 + 160\varepsilon_4 + 2\varepsilon_5,$$

where, with $R(dx) := |\xi(x, \mathcal{P}_\lambda; \rho)| \mathcal{P}_\lambda(dx)$, $N(C_x) := B_{10\rho}(x)$, and $p \in (2, \infty)$,

$$\begin{aligned} \varepsilon_3 &:= (\text{Var} W_\lambda(\rho))^{-p/2} \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx), \\ \varepsilon_4 &:= (\text{Var} W_\lambda(\rho))^{-p/2} \int_{Q_\lambda} \mathbb{E} R(N(C_x))^{p-1} \mathbb{E} R(dx), \\ \varepsilon_5 &:= (\text{Var} W_\lambda(\rho))^{-1/2} \sup_{x \in Q_\lambda} \mathbb{E} R(N(C_x)). \end{aligned}$$

We write $G_{x,\lambda} := \{D(x, \mathcal{P}_\lambda) \leq \rho\}$. For ε_3 , we have by definition of $R(dx)$ that

$$\begin{aligned} & \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\ &= \mathbb{E} \int_{Q_\lambda} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})| \mathbf{1}(G_{z,\lambda}) \mathcal{P}_\lambda(dz) \right)^{p-1} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathbf{1}(G_{x,\lambda}) \mathcal{P}_\lambda(dx) \\ &\leq \mathbb{E} \int_{Q_\lambda} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})| \mathcal{P}_\lambda(dz) \right)^{p-1} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx). \end{aligned}$$

Hölder's inequality $(\int_D |f| \mu(dx))^{p-1} \leq \int_D |f|^{p-1} \mu(dx) \cdot \mu(D)^{p-2}$ gives that

$$\begin{aligned} & \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\ &\leq \mathbb{E} \int_{Q_\lambda} \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \cdot \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx) \\ &\leq \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\ &\quad + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \setminus \{x\}} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx), \end{aligned}$$

where we write $\int_{N(C_x)} \cdots \mathcal{P}_\lambda(dz)$ as $\int_{\{x\}} \cdots \mathcal{P}_\lambda(dz) + \int_{N(C_x) \setminus \{x\}} \cdots \mathcal{P}_\lambda(dz)$. The inequality $|a||b|^{p-1} \leq |a|^p + |b|^p$ gives

$$\begin{aligned} & \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\ &\leq \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\ &\quad + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \setminus \{x\}} (|\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^p + |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p) \cdot \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(dx). \end{aligned}$$

Splitting the last integral into two integrals gives

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
& \leq \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \\
& \quad + \mathbb{E} \int_{Q_\lambda} \int_{N(C_x) \setminus \{x\}} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(dx) \\
& \quad + \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(N(C_x))^{p-1} \mathcal{P}_\lambda(dx) \\
& \leq \mathbb{E} \int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(dx) \\
& \quad + \mathbb{E} \iint_{0 < d(x,z) \leq 10\rho} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^p \mathcal{P}_\lambda(N(C_x))^{p-2} \mathcal{P}_\lambda(dx) \mathcal{P}_\lambda(dz) \\
& \quad + \mathbb{E} \int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-1} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(dx).
\end{aligned}$$

Now integrating the double integral gives

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
& \leq \mathbb{E} \int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-2} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(dx) \\
& \quad + \mathbb{E} \int_{Q_\lambda} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^p \cdot \mathcal{P}_\lambda(B_{20\rho}(z))^{p-1} \mathcal{P}_\lambda(dz) \\
& \quad + \mathbb{E} \int_{Q_\lambda} \mathcal{P}_\lambda(N(C_x))^{p-1} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})|^p \mathcal{P}_\lambda(dx).
\end{aligned}$$

Combining integrals and using Hölder's inequality for $p_1 \in (1, q/p)$ gives

$$\begin{aligned}
& \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\
& \leq 3 \mathbb{E} \int_{Q_\lambda} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^p \mathcal{P}_\lambda(B_{20\rho}(z))^{p-1} \mathcal{P}_\lambda(dz) \\
& \leq 3 \left\{ \mathbb{E} \int_{Q_\lambda} \mathcal{P}_\lambda(B_{20\rho}(z))^{\frac{(p-1)p_1}{p_1-1}} \mathcal{P}_\lambda(dz) \right\}^{\frac{p_1-1}{p_1}} \left\{ \mathbb{E} \int_{Q_\lambda} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{pp_1} \mathcal{P}_\lambda(dz) \right\}^{\frac{1}{p_1}} \quad (5.3)
\end{aligned}$$

Since $\mathcal{P}_\lambda^{\beta\Psi}$ is a Gibbs point process, we apply the Georgii-Nguyen-Zessin integral characterization of Gibbs point processes [24] to see that the conditional probability of observing an extra point of $\mathcal{P}_\lambda^{\beta\Psi}$ in the volume element dz , given that configuration without that point, equals $\exp(-\beta\Delta^\Psi(\{z\}, \mathcal{P}_\lambda^{\beta\Psi}))dz \leq dz$, where $\Delta^\Psi(\{z\}, \mathcal{P}_\lambda^{\beta\Psi})$ is defined at (1.2).

Using that $\mathbb{E} \mathcal{P}_\lambda^{\beta\Psi}(dx) \leq \tau dx$, we have from (5.3) that

$$\begin{aligned} & \mathbb{E} \int_{Q_\lambda} R(N(C_x))^{p-1} R(dx) \\ & \leq 3\tau \left\{ \mathbb{E} \int_{Q_\lambda} (\mathcal{P}_\lambda(B_{20\rho}(z)) + 1)^{\frac{(p-1)p_1}{p_1-1}} dz \right\}^{\frac{p_1-1}{p_1}} \left\{ \mathbb{E} \int_{Q_\lambda} |\xi(z, \mathcal{P}_\lambda \cup \{z\})|^{pp_1} dz \right\}^{\frac{1}{p_1}} \end{aligned} \quad (5.4)$$

Notice that $\mathcal{P}_\lambda(B_{20\rho}(x))$ is stochastically bounded by $\text{Po}(\tau M)$ with $M := \text{Vol}(B_{20\rho}(0))$, we have from Lemma 4.3 of [4] that $\mathbb{E} \{\mathcal{P}_\lambda(B_{20\rho}(x)) + 1\}^{(p-1)p_1/(p_1-1)} \leq c_1 \rho^{d(p-1)p_1/(p_1-1)}$, giving

$$\varepsilon_3 \leq 3\tau \text{Var}(W_\lambda(\rho))^{-p/2} c_1^{\frac{p_1-1}{p_1}} \rho^{d(p-1)} \lambda^{(p_1-1)/p_1} \left\{ \mathbb{E} \int_{Q_\lambda} |\xi(x, \mathcal{P}_\lambda \cup \{x\})|^{pp_1} dx \right\}^{\frac{1}{p_1}}.$$

Then since $w_{pp_1} \leq w_q$, we have

$$\varepsilon_3 \leq 3\tau \lambda \text{Var}(W_\lambda(\rho))^{-p/2} c_1^{\frac{p_1-1}{p_1}} w_q^p \rho^{d(p-1)}. \quad (5.5)$$

Next, we bound ε_4 . To this end, let $p_2 := pp_1/(p-1)$, we again replace the indicator function with 1 and then apply Hölder's inequality to get

$$\begin{aligned} & \int_{Q_\lambda} \mathbb{E} R(N(C_x))^{p-1} \mathbb{E} R(dx) \\ & = \int_{Q_\lambda} \mathbb{E} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})| \mathbf{1}(G_{z,\lambda}) \mathcal{P}_\lambda(dz) \right)^{p-1} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathbf{1}(G_{x,\lambda}) \mathcal{P}_\lambda(dx) \\ & \leq \int_{Q_\lambda} \mathbb{E} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})| \mathcal{P}_\lambda(dz) \right)^{p-1} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx) \\ & \leq \int_{Q_\lambda} \mathbb{E} \left\{ \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{p-1} \mathcal{P}_\lambda(dz) \mathcal{P}_\lambda(N(C_x))^{p-2} \right\} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx) \\ & \leq \int_{Q_\lambda} \mathbb{E} \left\{ \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{p-1} \mathcal{P}_\lambda(B_{20\rho}(z))^{p-2} \mathcal{P}_\lambda(dz) \right\} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx) \\ & \leq \int_{Q_\lambda} \left\{ \mathbb{E} \int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})|^{p_2(p-1)} \mathcal{P}_\lambda(dz) \right\}^{\frac{1}{p_2}} \\ & \quad \left\{ \mathbb{E} \int_{N(C_x)} \mathcal{P}_\lambda(B_{20\rho}(z))^{(p-2)\frac{p_2}{p_2-1}} \mathcal{P}_\lambda(dz) \right\}^{\frac{p_2-1}{p_2}} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx). \end{aligned} \quad (5.6)$$

Reasoning as for (5.4), we obtain from (5.6) that

$$\begin{aligned}
& \int_{Q_\lambda} \mathbb{E} R(N(C_x))^{p-1} \mathbb{E} R(dx) \\
& \leq \int_{Q_\lambda} \left\{ \int_{N(C_x)} \mathbb{E} |\xi(z, \mathcal{P}_\lambda \cup \{z\})|^{p_2(p-1)} \tau dz \right\}^{\frac{1}{p_2}} \\
& \quad \left\{ \int_{N(C_x)} \mathbb{E} (\mathcal{P}_\lambda(B_{20\rho}(z)) + 1)^{(p-2)\frac{p_2}{p_2-1}} \tau dz \right\}^{\frac{p_2-1}{p_2}} \mathbb{E} |\xi(x, \mathcal{P}_\lambda \setminus \{x\})| \mathcal{P}_\lambda(dx) \\
& \leq \tau^2 w_{pp_1}^{p-1} c_2^{\frac{p_2-1}{p_2}} \rho^{d(p-2)} \int_{Q_\lambda} \left\{ \int_{N(C_x)} dz \right\}^{\frac{1}{p_2}} \left\{ \int_{N(C_x)} dz \right\}^{\frac{p_2-1}{p_2}} w_{pp_1} dx \\
& \leq w_{pp_1}^p c_3 \lambda \rho^{d(p-1)}.
\end{aligned}$$

Hence

$$\varepsilon_4 \leq (\text{Var} W_\lambda(\rho))^{-p/2} w_q^p c_3 \lambda \rho^{d(p-1)}, \quad (5.7)$$

showing that the bounds for ε_3 and ε_4 coincide. Turning to ε_5 , we have

$$\begin{aligned}
\varepsilon_5 & \leq (\text{Var} W_\lambda(\rho))^{-1/2} \sup_{x \in Q_\lambda} \mathbb{E} \left(\int_{N(C_x)} |\xi(z, \mathcal{P}_\lambda \setminus \{z\})| \mathcal{P}_\lambda(dz) \right) \\
& \leq (\text{Var} W_\lambda(\rho))^{-1/2} \sup_{x \in Q_\lambda} \left(\int_{N(C_x)} \mathbb{E} |\xi(z, \mathcal{P}_\lambda \cup \{z\})| \tau dz \right) \\
& \leq \text{Var}(W_\lambda(\rho))^{-1/2} \sup_{x \in Q_\lambda} \left(\int_{N(C_x)} \{ \mathbb{E} |\xi(z, \mathcal{P}_\lambda \cup \{z\})|^{pp_1} \}^{\frac{1}{pp_1}} \tau dz \right) \\
& \leq \text{Var}(W_\lambda(\rho))^{-1/2} w_q c_4 \rho^d.
\end{aligned} \quad (5.8)$$

Combining estimates (5.5), (5.7) and (5.8), we get (5.1).

Assuming condition (1.6), using (4.3) with $G = \emptyset$ and Theorem 1.1, we have $\text{Var}[W_\lambda(\rho)] \geq c_5 \lambda$. When $p = 3$, this, together with (5.1), gives (5.2).

To complete the proof, we need to replace $W_\lambda(\rho)$ with W_λ . We rely heavily on Lemma 4.2 for this. Note for all $\epsilon_1 \in \mathbb{R}$ and $\epsilon_2 > -0.6$,

$$\begin{aligned}
d_K(N(0, 1), N(\epsilon_1, 1 + \epsilon_2)) & \leq d_K(N(0, 1), N(\epsilon_1, 1)) + d_K(N(\epsilon_1, 1), N(\epsilon_1, 1 + \epsilon_2)) \\
& \leq \frac{|\epsilon_1|}{\sqrt{2\pi}} + \frac{|\epsilon_2|}{\sqrt{2e\pi}}.
\end{aligned} \quad (5.9)$$

Now $d_K(X, N(0, 1)) = d_K(aX, N(0, a^2)) = d_K(aX + b, N(b, a^2))$ holds for X with $\mathbb{E} X = 0$ and all constants a and b . Hence

$$\begin{aligned}
& d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1) \right) = d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}}, N \left(\frac{\mathbb{E} W_\lambda - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}}, \frac{\text{Var} W_\lambda}{\text{Var} W_\lambda(\rho)} \right) \right) \\
& \leq d_K \left(\frac{W_\lambda - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}}, N(0, 1) \right) + d_K \left(N(0, 1), N \left(\frac{\mathbb{E} W_\lambda - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}}, \frac{\text{Var} W_\lambda}{\text{Var} W_\lambda(\rho)} \right) \right)
\end{aligned} \quad (5.10)$$

by the triangle inequality for d_K . Now for any random variables Y and Y' we have

$$d_K(Y, N(0, 1)) \leq d_K(Y', N(0, 1)) + \mathbb{P}[Y \neq Y'] \quad (5.11)$$

which follows from $|\mathbb{P}[Y \leq t] - \Phi(t)| \leq |\mathbb{P}[Y' \leq t] - \Phi(t)| + |\mathbb{P}[Y' \leq t] - \mathbb{P}[Y \leq t]|$. We have by (5.11) and (5.9) that

$$\begin{aligned} & d_K\left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1)\right) \\ & \leq \mathbb{P}[W_\lambda \neq W_\lambda(\rho)] + d_K\left(\frac{W_\lambda(\rho) - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}}, N(0, 1)\right) \\ & \quad + \frac{1}{\sqrt{2\pi}} \left| \frac{\mathbb{E} W_\lambda - \mathbb{E} W_\lambda(\rho)}{\sqrt{\text{Var} W_\lambda(\rho)}} \right| + \frac{1}{\sqrt{2e\pi}} \left| \frac{\text{Var} W_\lambda - \text{Var} W_\lambda(\rho)}{\text{Var} W_\lambda(\rho)} \right|. \end{aligned} \quad (5.12)$$

However, the Cauchy-Schwarz inequality ensures

$$|\mathbb{E} W_\lambda - \mathbb{E} W_\lambda(\rho)| \leq \|W_\lambda - W_\lambda(\rho)\|_2 \mathbb{P}(W_\lambda \neq W_\lambda(\rho))^{1/2} \leq \lambda^{-1},$$

where the last inequality is due to (4.1), (3.6) and the arbitrariness of L . Hence, it follows from (5.12) that

$$d_K\left(\frac{W_\lambda - \mathbb{E} W_\lambda}{\sqrt{\text{Var} W_\lambda}}, N(0, 1)\right) \leq \lambda^{-2} + O((\text{Var} W_\lambda)^{-p/2} \lambda (\ln \lambda)^{d(p-1)}),$$

where we use (3.6) with $L = 2$, (5.1) and (4.3) with $G = \emptyset$. \square

Proof of Theorem 1.3. The bound (1.22) follows from Lemma 4.4 and Lemma 4.2(b) with $G = \emptyset$. The proof of (1.21) follows by replacing Q_λ with \tilde{S}_λ in the proof of (1.18), whereas (1.23) follows by combining (1.21) and (1.22). \square

Acknowledgements. We thank Yogeshwaran Dhandapani for showing us Lemma 4.6, which essentially first appeared in [6]. J. Yukich gratefully acknowledges generous and kind support from the Department of Mathematics and Statistics at the University of Melbourne, where this work was initiated.

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